

From Chirality to the Standard Model

A Uniqueness and No-Go Theorem for Compact Lie-Algebra Embeddings

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github.com/emad-ii/chirality-standard-model

Abstract

An inclusion of compact Lie algebras supplies its own representation: the adjoint complement $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$. We classify effective proper inclusions $\mathfrak{h} \subsetneq \mathfrak{g}$ for which this real module is irreducible of complex type and the cubic trace tensor of one complex half has nonzero radical. Effectivity means that no nonzero ideal of $\mathfrak{g}$ is contained in $\mathfrak{h}$. Up to isomorphism of Lie-algebra pairs and duality, the unique solution is

$$\mathfrak{e}_8 \supset \mathfrak{e}_6 \oplus \mathfrak{su}(3)_M, \quad V = \mathbf{27} \boxtimes \mathbf{3}, \quad \text{Rad}(c_V) = \mathfrak{e}_6.$$

Every other pair in the stated complex-type domain has zero radical. The proof covers arbitrary rank: toral gradings enumerate the equal-rank cases, and index arithmetic and highest-weight arguments exclude the lower-rank cases. The hypotheses also imply a realization as a compact homogeneous space with closed isotropy subgroup. The commuting $SU(3)_M$ factor gives three copies of the $\mathbf{27}$ on restriction to E_6 .

Inside this E_6 , a second classification concerns the 38,880 triples formed from its three Weyl orbits of maximal proper closed root subsystems. There are six orbits of triples. Exactly two have an irreducible A_2 component in their common intersection; these are the two extremal orbits. In every triple, one pairwise derived group has strictly greatest dimension. Using that group in the pair-first construction, compatible representatives of the extremal orbits give

$$G_{\text{SM}} \subset G_{\text{LR}} \subset G_{\text{PS}} \subset \text{Spin}(10) \subset E_6^{\text{sc}}, \quad Y = T_R^3 + \frac{B-L}{2},$$

including their faithful global forms and charge lattices. The full restriction is

$$V|_{G_{\text{SM}}} \cong 3\mathcal{F}_{\text{SM}} \oplus 3(\mathbf{5} \oplus \bar{\mathbf{5}})|_{G_{\text{SM}}} \oplus 6\mathbf{1}.$$

Here $\mathcal{F}_{\text{SM}}$ is one minimal chiral family. The net chiral class is $3\chi(\mathcal{F}_{\text{SM}})$, up to sign; the other summands are vectorlike or singlets.

1 Introduction

Matter in the Standard Model is chiral: left- and right-handed fermions transform differently under the weak interaction. A consistent chiral gauge theory must also satisfy anomaly cancellation [1]. These requirements constrain the matter representations. Can a mathematical formulation of them go further, and determine an internal symmetry and its matter representation together?

An inclusion of compact real Lie algebras $\mathfrak{h} \subsetneq \mathfrak{g}$ provides a precise algebraic question. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ induces a representation on the quotient,

$$\mathfrak{q} = \mathfrak{g}/\mathfrak{h}.$$

We call this module the *adjoint complement*. It belongs to the embedding: changing the pair changes the algebra and its representation together. We study which pairs satisfy a complex-type condition and a condition on the cubic trace tensor, without first specifying an exceptional group or a matter multiplet.

The classification problem. We consider all effective proper inclusions of finite-dimensional compact real Lie algebras: no nonzero ideal of $\mathfrak{g}$ is contained in $\mathfrak{h}$.

The Killing–Cartan classification exhausts the finite-dimensional compact simple Lie algebras: the four classical series A_n, B_n, C_n, D_n and the five exceptional types G_2, F_4, E_6, E_7, E_8 [13, Ch. VI]. A *simple* Lie algebra is nonabelian and has no nonzero proper ideal; a *semisimple* Lie algebra is a finite direct sum of simple Lie algebras. Every finite-dimensional compact real Lie algebra is a direct sum of a semisimple algebra and an abelian centre. Thus the starting domain includes all compact algebra types, with no rank cutoff; the subsequent argument determines which embeddings satisfy the two conditions.

The first condition is

$$\text{End}_{\mathfrak{h}}(\mathfrak{q}) \cong \mathbb{C}.$$

It implies that the real module $\mathfrak{q}$ is irreducible and has an invariant complex structure, unique up to sign. Its complexification is $V \oplus V^*$, where V and V^* are inequivalent irreducible representations. Thus irreducibility and complex type are encoded in one condition, rather than imposed separately.

The second condition concerns the cubic trace tensor c_V on $\mathfrak{h}$. A generator lies in its radical when inserting that generator makes the cubic vanish, whatever the other two arguments are. This radical is automatically an ideal, and we require it to be nonzero:

$$\text{Rad}(c_V) \neq 0.$$

The test includes all mixed insertions from $\mathfrak{h}$. It is in general stronger than requiring the cubic to vanish only on a separately specified gauge subalgebra. The ideal on which the cubic vanishes is determined by the representation itself.

One pair remains. Under these conditions, the pair is unique up to isomorphism of Lie-algebra pairs and duality:

$$\mathfrak{e}_8 \supset \mathfrak{e}_6 \oplus \mathfrak{su}(3)_M, \quad V = \mathbf{27} \boxtimes \mathbf{3}.$$

Here $\boxtimes$ denotes the tensor product for the two simple factors. The cubic radical is $\mathfrak{e}_6$; its commuting factor $\mathfrak{su}(3)_M$ acts on a three-dimensional multiplicity space. Restriction to E_6 therefore gives three copies of the $\mathbf{27}$. The embedding and the multiplicity are conclusions of the classification.

The theorem is also an exclusion result. Every inequivalent pair satisfying the standing algebraic hypotheses and the complex-type condition has zero cubic radical. Proving that the displayed pair works is therefore only one part of the argument. We must exclude every other equal-rank and lower-rank possibility, without a rank cutoff.

Where homogeneity enters. The hypotheses imply that $\mathfrak{g}$ is simple and $\mathfrak{h}$ is semisimple. They therefore integrate to a compact group G with a connected closed subgroup H ; the tangent space of G/H at the base point is precisely $\mathfrak{g}/\mathfrak{h}$. Homogeneity is the geometric realization of the algebraic pair, not an additional assumption about physical spacetime. Section 2 proves this passage. The substantive restriction is that the representation is the entire adjoint complement and is irreducible of complex type.

Relation to earlier constructions. Ordinary $SU(5)$, $Spin(10)$, and E_6 unification organizes a family in $\mathbf{10} \oplus \bar{\mathbf{5}}$, $\mathbf{16}$, or $\mathbf{27}$, but does not by itself fix how many times it is repeated [19, 20, 21]. Homogeneous representations have also been used in coset-space dimensional reduction and exceptional family unification [2, 3, 4, 5, 6, 7, 8]. Heterotic standard embeddings relate chirality to compactification data [9], and recent work studies anomalies of exceptional family-unification cosets through bordism [25]. These constructions provide precedents for tying matter to geometry. The problem here is the converse: classify all compact Lie-algebra pairs satisfying the two invariant conditions.

Anomaly cancellation has long been used to constrain matter content rather than merely to check an existing model [11, 12]. There are also exhaustive methods for finding semisimple completions of a supplied gauge algebra and fermion representation [27], and systematic studies of which anomaly-free spectra admit unification [28]. Our input differs: the representation is $\mathfrak{g}/\mathfrak{h}$, and both the pair and its canonical representation vary.

Intersections and matter. A second problem begins inside the resulting E_6 . We form triples containing one maximal proper closed root subsystem from each of its three Weyl orbits. The 38,880 triples form six orbits. We classify both the triples with largest pairwise intersections and those with largest common intersection. Exactly these two extremal orbits have a common irreducible A_2 component.

We then compare the dimensions of the intermediate semisimple groups obtained from pairwise intersections. For every triple, the $D_5-(A_5 + A_1)$ pairing gives the unique largest group. This ordering uses only the root intersections. Intersecting that group with the third regular group gives the Standard Model and left-right groups on the two extremal orbits. Compatible representatives give their inclusion in a common Pati-Salam chain, with its finite central quotients and hypercharge embedding. Restricting the $\mathbf{27}$ to the selected intermediate group then gives chiral dimension 15 or 16. The branchings and the familiar intersection $SU(5) \cap G_{PS}$ are classical [22, 23, 24]; the two exhaustive classifications establish which configurations realize these branchings.

Finally, restriction gives the complete matter decomposition. Each $\mathbf{27}$ contains one minimal chiral Standard Model family, a vectorlike pair, and two singlets. The multiplicity factor supplies three copies. Section 5 keeps this full representation separate from its net chiral class.

Section 2 defines the two invariant conditions and states the unique-pair theorem. Section 3 proves the exhaustive Lie-algebra census. Section 4 classifies the finite E_6 triple space and reconstructs the faithful subgroup chains. Section 5 restricts the selected module and computes its chiral index and anomalies. The finite algorithms and their completeness arguments are explained where the proofs use them. The source, certificates, and verification instructions are identified in the computational-material statement at the end of the paper.

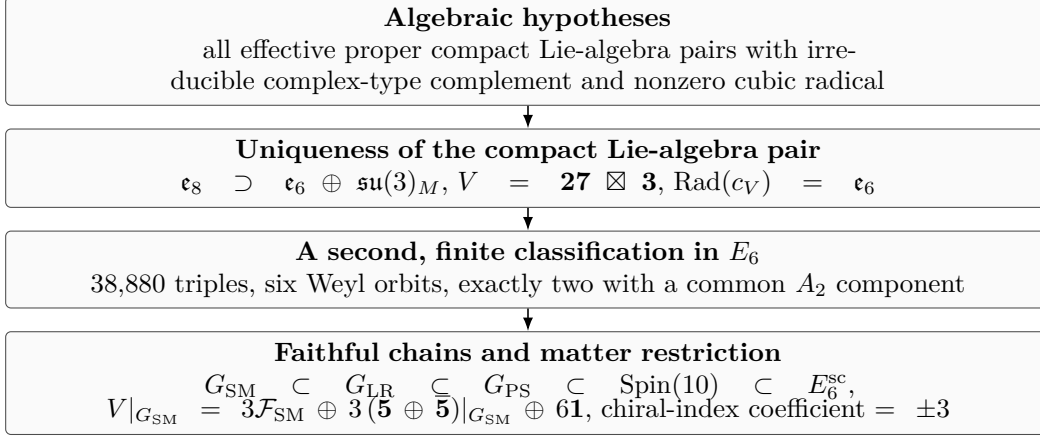


Figure 1: The two classification problems. The subgroup construction uses the two extremal orbits and the unique largest pairwise derived group. The last coefficient records the net chiral class of the restricted module.

2 Adjoint complements and cubic anomalies

We first define complex-type complements and the cubic radical. These two conditions specify the Lie-algebra classification problem.

Let $\mathfrak{g}$ be a finite-dimensional compact real Lie algebra and let $\mathfrak{h} \subsetneq \mathfrak{g}$ be a Lie subalgebra. Here compactness means that $\mathfrak{g}$ admits a positive-definite ad-invariant inner product; semisimplicity is not assumed. Write $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$ for the quotient module and assume *effectivity*: $\mathfrak{h}$ contains no nonzero ideal of $\mathfrak{g}$. A positive-definite $\text{ad}(\mathfrak{g})$ -invariant inner product gives

$$\mathfrak{q} = \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{h}^\perp, \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad [\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}.$$

We identify the quotient with this orthogonal complement. The skew-adjoint action of $\mathfrak{h}$ makes the module completely reducible. The pair itself fixes the underlying module as $\mathfrak{g}/\mathfrak{h}$.

Fraktur letters denote Lie algebras. For a representation R , R^* denotes its dual and $\bar{R}$ its complex conjugate; for compact groups these are isomorphic. We use R^* in duality statements and bars in conventional representation labels. The symbol $\boxtimes$ denotes an external tensor product, and a superscript “sc” denotes the simply connected compact group.

2.1 Complex-type complements

The algebra $\text{End}_{\mathfrak{h}}(\mathfrak{q})$ consists of the real linear maps on $\mathfrak{q}$ that commute with $\mathfrak{h}$. We require

$$\boxed{\text{End}_{\mathfrak{h}}(\mathfrak{q}) \cong \mathbb{C}}. \tag{1}$$

Compactness makes $\mathfrak{q}$ completely reducible. If it had a proper invariant summand, projection onto that summand would be a nontrivial idempotent in $\text{End}_{\mathfrak{h}}(\mathfrak{q})$. Since $\mathbb{C}$ has no such idempotent, $\mathfrak{q}$ is irreducible. Choose J corresponding to $i \in \mathbb{C}$. Then $J^2 = -I$, and J is unique up to sign. Its $+i$ - and $-i$ -eigenspaces in $\mathfrak{q}_{\mathbb{C}}$ are irreducible dual $\mathfrak{h}$ -modules V and V^* , so

$$\mathfrak{q}_{\mathbb{C}} = V \oplus V^*, \quad V \not\cong V^*.$$

The two choices J and $-J$ exchange V and V^* ; we call them the two complex orientations. The Lie-algebra pair determines the unordered dual pair $\{V, V^*\}$, not a preferred member.

2.2 The cubic anomaly and its radical

The inner product on $\mathfrak{g}$ induces an $\mathfrak{h}$ -invariant Hermitian inner product on V . The operators $\rho_V(X)$ are then skew-Hermitian, so $T_X = -i\rho_V(X)$ is Hermitian. With $\{A, B\} = AB + BA$, define

$$\begin{aligned} c_V(X, Y, Z) &= \frac{1}{2} \text{Tr}_V(T_X\{T_Y, T_Z\}), \\ \Phi_V(X) &= c_V(X, \cdot, \cdot), \quad \text{Rad}(c_V) = \ker \Phi_V. \end{aligned} \tag{2}$$

For a left-handed Weyl multiplet in four spacetime dimensions, c_V is the group factor in the local triangle anomaly. Equivalently, it polarizes the cubic trace underlying the pure-gauge term $\text{Tr}_V(F^3)$ in the six-form anomaly polynomial [10]. The map $\Phi_V : \mathfrak{h} \rightarrow \text{Sym}^2 \mathfrak{h}^*$ is equivariant, so its kernel is invariant under the adjoint action and is therefore an ideal. It is the largest ideal $\mathfrak{a} \triangleleft \mathfrak{h}$ for which every cubic with one argument in $\mathfrak{a}$ vanishes. Equivalently,

$$\exists 0 \neq \mathfrak{k} \triangleleft \mathfrak{h} : c_V(\mathfrak{k}, \mathfrak{h}, \mathfrak{h}) = 0 \iff \text{Rad}(c_V) \neq 0.$$

We require

$$\text{Rad}(c_V) \neq 0. \tag{3}$$

This is the second invariant condition. For a particular nonzero ideal $\mathfrak{k}$, it is in general stronger than vanishing only on $\mathfrak{k}^3$:

$$\begin{aligned} \text{vanishing only on } \mathfrak{k}^3 & : c_V(\mathfrak{k}, \mathfrak{k}, \mathfrak{k}) = 0, \\ \text{vanishing with one } \mathfrak{k}\text{-argument} & : c_V(\mathfrak{k}, \mathfrak{h}, \mathfrak{h}) = 0. \end{aligned} \tag{4}$$

When $\mathfrak{h}$ is semisimple, mixed cubics between distinct simple ideals vanish, and the two conditions agree for a fixed ideal $\mathfrak{k}$. They need not agree when $\mathfrak{h}$ has a centre. The radical formulation treats both cases uniformly and does not presuppose a choice of ideal. Every ideal satisfying $c_V(\mathfrak{k}, \mathfrak{h}, \mathfrak{h}) = 0$ lies in $\text{Rad}(c_V)$. Hence every component with one argument in the radical vanishes, including components with arbitrary $\mathfrak{h}$ -valued background insertions. Local counterterms can redistribute the consistent anomaly among currents, but they do not alter the invariant polynomial represented by c_V [10]. Condition (3) applies to that invariant polynomial, not merely to a chosen subalgebra. Duality sends c_V to $-c_V$ and leaves its radical unchanged.

Theorem 2.1 (Uniqueness of the compact Lie-algebra pair¹). *Among the effective proper compact Lie-algebra pairs specified above, conditions (1) and (3) have, up to isomorphism of Lie-algebra pairs and exchange of V with V^* , the unique solution*

$$(\mathfrak{g}, \mathfrak{h}, \text{Rad}(c_V), \{V, V^*\}) = (\mathfrak{e}_8, \mathfrak{e}_6 \oplus \mathfrak{su}(3)_M, \mathfrak{e}_6, \{\mathbf{27} \boxtimes \mathbf{3}, \overline{\mathbf{27}} \boxtimes \overline{\mathbf{3}}\}). \tag{5}$$

Conversely, this pair satisfies the hypotheses.

Remark (Contrapositive and no-go consequence). Every other effective proper compact Lie-algebra pair with irreducible complex-type complement has $\text{Rad}(c_V) = 0$. No candidate in the full census has $\text{Rad}(c_V) = \mathfrak{h}$; the survivor is the unique case with a proper nonzero radical. Equivalently, any different pair must fail at least one standing hypothesis or one of conditions (1) and (3). Group-level global forms are determined separately in Section 4.

Remark (Cubic radical and multiplicity factor). The pair fixes $\mathfrak{h}$ and $\mathfrak{g}$ simultaneously. In the surviving case the cubic radical then separates $\mathfrak{h} = \mathfrak{e}_6 \oplus \mathfrak{su}(3)_M$ into the cubic-anomaly-safe ideal $\mathfrak{e}_6$ and the commuting multiplicity factor $\mathfrak{su}(3)_M$.

¹For my son Noah, on his eighteenth birthday.

Remark (Invariant degrees and the surviving radical). The basic invariant degrees of $W(E_6)$ are 2, 5, 6, 8, 9, 12. The cubic trace polynomial of any E_6 -representation therefore vanishes on a Cartan subalgebra, and cubic gauge anomalies remain zero after restriction to subgroups of E_6 . By contrast, $W(A_2)$ has invariant degrees 2, 3. For $V = \mathbf{27} \boxtimes \mathbf{3}$, the nonzero cubic lies in the $SU(3)_M$ direction, while every component with an E_6 insertion vanishes. This difference in invariant degrees accounts for $\text{Rad}(c_V) = \mathfrak{e}_6$.

Lemma 2.2 (Structural reduction of the algebraic domain). *Under the standing pair hypotheses and Conditions (1) and (3), $\mathfrak{q}$ is an irreducible real $\mathfrak{h}$ -module, the quotient action is faithful, $\mathfrak{g}$ is simple, and $\mathfrak{h}$ is semisimple and maximal among proper Lie subalgebras of $\mathfrak{g}$.*

Proof. Maximality. Irreducibility was proved after (1). An intermediate algebra $\mathfrak{h} \subsetneq \mathfrak{l} \subsetneq \mathfrak{g}$ would then make $\mathfrak{l}/\mathfrak{h}$ a proper nonzero $\mathfrak{h}$ -submodule of $\mathfrak{q}$. Hence $\mathfrak{h}$ is maximal among proper subalgebras of $\mathfrak{g}$.

Faithfulness. Let $\mathfrak{k}_0$ be the kernel of the quotient representation. Since $\mathfrak{k}_0$ acts trivially on $\mathfrak{g}/\mathfrak{h}$, $[\mathfrak{k}_0, \mathfrak{q}] \subseteq \mathfrak{h}$. But $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}$, so this bracket lies in both summands and vanishes. Since $\mathfrak{k}_0 \triangleleft \mathfrak{h}$, this makes $\mathfrak{k}_0$ an ideal of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ contained in $\mathfrak{h}$. Effectivity therefore gives $\mathfrak{k}_0 = 0$. Because an element acting trivially on V also acts trivially on V^* , the $\mathfrak{h}$ -module V is faithful.

Simplicity of $\mathfrak{g}$. The projection of $\mathfrak{z}(\mathfrak{g})$ to $\mathfrak{q}$ is fixed by $\mathfrak{h}$. If it were nonzero, irreducibility would make all of $\mathfrak{q}$ fixed. Every subspace would then be invariant, forcing $\dim \mathfrak{q} = 1$; its endomorphism algebra would be $\mathbb{R}$, contrary to (1). It follows that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{h}$. It is an ideal of $\mathfrak{g}$, so effectivity gives $\mathfrak{z}(\mathfrak{g}) = 0$; compactness makes $\mathfrak{g}$ semisimple.

Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ is simple and $\mathfrak{g}_2 \neq 0$ is the sum of the remaining simple ideals. Effectivity implies that neither ideal is contained in $\mathfrak{h}$. Since $\mathfrak{h}$ is maximal and each $\mathfrak{g}_i$ is an ideal,

$$\mathfrak{h} + \mathfrak{g}_1 = \mathfrak{h} + \mathfrak{g}_2 = \mathfrak{g}.$$

If $X \in \mathfrak{h} \cap \mathfrak{g}_1$, write any $Y \in \mathfrak{g}$ as $Y = Y_{\mathfrak{h}} + Y_2$, with $Y_{\mathfrak{h}} \in \mathfrak{h}$ and $Y_2 \in \mathfrak{g}_2$. Then $[X, Y] = [X, Y_{\mathfrak{h}}] \in \mathfrak{h}$, so X acts trivially on $\mathfrak{g}/\mathfrak{h}$. Faithfulness gives $\mathfrak{h} \cap \mathfrak{g}_1 = 0$, and similarly $\mathfrak{h} \cap \mathfrak{g}_2 = 0$. If $p_i : \mathfrak{g} \rightarrow \mathfrak{g}_i$ are the projections, the sum identities give surjectivity and the intersection identities give injectivity. The restrictions $p_i|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{g}_i$ are therefore isomorphisms. Since $\mathfrak{g}_1$ is simple, $\mathfrak{h}$ is simple, and so is $\mathfrak{g}_2$. After an automorphism the pair is $(\mathfrak{l} \oplus \mathfrak{l}, \text{diag } \mathfrak{l})$. Its quotient is the adjoint $\mathfrak{l}$ -module, whose real endomorphism algebra is $\mathbb{R}$, again contradicting (1). This contradiction proves that $\mathfrak{g}$ is simple.

Semisimplicity of $\mathfrak{h}$. For $0 \neq z \in \mathfrak{z}(\mathfrak{h})$, Schur's lemma gives $T_z = \lambda I$ for some $\lambda \in \mathbb{R}$, and faithfulness gives $\lambda \neq 0$. For $0 \neq X \in \text{Rad}(c_V)$, faithfulness gives $T_X \neq 0$, while equation (2) would give

$$0 = c_V(X, z, X) = \lambda \text{Tr}_V(T_X^2),$$

which is impossible because the nonzero Hermitian operator T_X satisfies $\text{Tr}_V(T_X^2) > 0$. Consequently $\mathfrak{z}(\mathfrak{h}) = 0$; compactness then makes $\mathfrak{h}$ semisimple. $\square$

Remark (Homogeneous realization). No closed-subgroup assumption is needed in the algebraic formulation. By Lemma 2.2, both $\mathfrak{g}$ and $\mathfrak{h}$ are compact semisimple Lie algebras. Let G^{sc} and H^{sc} be their simply connected compact groups. The inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ integrates to a homomorphism $H^{\text{sc}} \rightarrow G^{\text{sc}}$. Its image H is connected and compact, hence closed, and has Lie algebra $\mathfrak{h}$. Thus G^{sc}/H is an infinitesimally effective compact homogeneous space whose isotropy module is $\mathfrak{g}/\mathfrak{h}$. Conversely, every connected, infinitesimally effective compact homogeneous pair gives the corresponding effective Lie-algebra pair. Under (1) and (3), the two formulations therefore classify exactly the same Lie-algebra pairs; global central quotients are separate data.

Remark (Strength of the mixed condition). The mixed arguments are essential. If only $c_V(\mathfrak{k}, \mathfrak{k}, \mathfrak{k}) = 0$ were imposed, the pairs $\mathfrak{e}_6 \supset \mathfrak{so}(10) \oplus \mathfrak{u}(1)$ and $\mathfrak{e}_7 \supset \mathfrak{e}_6 \oplus \mathfrak{u}(1)$ would satisfy it for $\mathfrak{k} = \mathfrak{so}(10)$ and $\mathfrak{k} = \mathfrak{e}_6$, respectively; neither satisfies (3).

3 Exhaustive census of compact Lie-algebra pairs

By Lemma 2.2, $\mathfrak{g}$ is compact and simple, $\mathfrak{h}$ is maximal semisimple, and the complement is irreducible of complex type. We separate the remaining pairs by rank. Full-rank pairs reduce to a finite root-grading calculation. For smaller-rank $\mathfrak{h}$, index arithmetic excludes the exceptional ambient algebras, and a uniform highest-weight argument excludes the classical ones.

We use $A_n = \mathfrak{su}(n+1)$. A subalgebra is called regular when it is normalised by a Cartan subalgebra of $\mathfrak{g}$.

3.1 The full-rank branch

A regular maximal subalgebra has full rank. Indeed, if $\text{rank } \mathfrak{h} < \text{rank } \mathfrak{g}$, regularity supplies a nonzero toral subalgebra $\mathfrak{t}_0 \subset C_{\mathfrak{g}}(\mathfrak{h})$, and

$$\mathfrak{h} \subsetneq \mathfrak{h} + \mathfrak{t}_0 \subsetneq \mathfrak{g},$$

contrary to maximality. Conversely, any equal-rank subalgebra contains a maximal torus of $\mathfrak{g}$ and is regular.

Lemma 3.1 (Saturation of a closed root subsystem). *If $\Psi \subseteq \Phi$ is a closed root subsystem, then*

$$\Phi \cap \mathbb{Z}\Psi = \Psi.$$

In particular, if $\Psi \subsetneq \Phi$, then $\mathbb{Z}\Psi \neq \mathbb{Z}\Phi$.

Proof. Let $\alpha \in \Phi \cap \mathbb{Z}\Psi$. Write $\alpha = \beta_1 + \cdots + \beta_m$, with each $\beta_j \in \Psi$, and choose such an expression with m minimal; negative coefficients may be absorbed by replacing roots of Ψ with their negatives. The claim is immediate for $m = 1$. If $m > 1$, then

$$(\alpha, \alpha) = \sum_{j=1}^m (\alpha, \beta_j) > 0,$$

so $(\alpha, \beta_j) > 0$ for some j . If $\alpha = \beta_j$, then $\alpha \in \Psi$. Otherwise the root-string property gives $\alpha - \beta_j \in \Phi$. This root lies in $\mathbb{Z}\Psi$ and is a sum of $m - 1$ roots of Ψ , hence belongs to Ψ by induction. Closedness of Ψ then gives $\alpha = (\alpha - \beta_j) + \beta_j \in \Psi$. Thus $\Phi \cap \mathbb{Z}\Psi \subseteq \Psi$; the reverse inclusion is immediate. If $\mathbb{Z}\Psi = \mathbb{Z}\Phi$, the displayed equality gives $\Phi = \Psi$, proving the final assertion. $\square$

Lemma 3.2 (Complex type and cyclic root gradings). *Suppose $\text{rank } \mathfrak{h} = \text{rank } \mathfrak{g}$. Then the root spaces of $\mathfrak{g}_{\mathbb{C}}$ relative to a common Cartan subalgebra admit one of the following two gradings:*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{g}_1 \quad (\mathbb{Z}_2), \quad \text{or} \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad (\mathbb{Z}_3),$$

where in the second case $\mathfrak{g}_1 \cong V$, $\mathfrak{g}_2 \cong V^$, and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. The involutive case has a self-conjugate quotient and therefore cannot satisfy condition (1).*

Proof. Let Φ be the root system of $\mathfrak{g}_{\mathbb{C}}$, let Ψ be the root subsystem of $\mathfrak{h}_{\mathbb{C}}$, and put $Q = \mathbb{Z}\Phi$, $L = \mathbb{Z}\Psi$. Root operators of $\mathfrak{h}$ change weights by elements of L . Since V is irreducible, all roots whose root spaces lie in V have one class $c \in Q/L$; those in V^* have class $-c$. Every root therefore has class

0, c , or $-c$, and these classes generate Q/L . Since Ψ is proper and closed, Lemma 3.1 gives $Q/L \neq 0$. Hence $c \neq 0$.

If $2c = 0$, the quotient is the odd part of an inner involution. Compact conjugation preserves that odd root set because $-c = c$. For a classical ambient algebra, an inner involution either has a unitary block centralizer with nonzero center, or has an orthogonal or symplectic block centralizer whose odd module is a self-dual tensor product. For an exceptional ambient algebra, the finite involutive calculation described in Proposition 3.3 shows that whenever the fixed algebra is semisimple, the odd weight graph is connected. In every case compatible with semisimplicity the irreducible odd module is self-conjugate, not of complex type.

Assume now that $2c \neq 0$. If $[V, V] = 0$, compact conjugation also gives $[V^*, V^*] = 0$, while the root classes give $[V, V^*] \subseteq \mathfrak{h}_{\mathbb{C}}$. The map acting as 0, +1, -1 on $\mathfrak{h}_{\mathbb{C}}, V, V^*$, respectively, is then a nonzero derivation of the simple algebra $\mathfrak{g}_{\mathbb{C}}$. It is inner and centralizes $\mathfrak{h}_{\mathbb{C}}$. But $\mathfrak{h}$ is semisimple of full rank, so its roots span the dual of the common Cartan and its centralizer in $\mathfrak{g}_{\mathbb{C}}$ is zero, a contradiction. Hence $[V, V] \neq 0$. A nonzero bracket has root class $2c$, which must be one of 0, c , $-c$. The first possibility is excluded by $2c \neq 0$, and the second would give $c = 0$. Thus $2c = -c$, so $3c = 0$. The quotient map $Q \rightarrow Q/L \cong \mathbb{Z}_3$ defines a toral order-three automorphism with fixed algebra $\mathfrak{h}_{\mathbb{C}}$ and the asserted grading. $\square$

Proposition 3.3 (Direct toral grading census). *Let $(\mathfrak{g}, \mathfrak{h})$ satisfy the standing pair hypotheses and Conditions (1) and (3), with $\text{rank } \mathfrak{h} = \text{rank } \mathfrak{g}$. The pair must be isomorphic to exactly one of the six candidates below. Here condition (3) has been used through Lemma 2.2 to establish semisimplicity of $\mathfrak{h}$; the cubic coefficients are evaluated in Table 1.*

$$\mathfrak{g}_2 \supset A_2, \quad \mathfrak{f}_4 \supset A_2 + A_2, \quad \mathfrak{e}_6 \supset A_2^3, \quad \mathfrak{e}_7 \supset A_5 + A_2, \quad \mathfrak{e}_8 \supset A_8, \quad \mathfrak{e}_8 \supset E_6 + A_2.$$

Their complex halves have dimensions

$$3, \quad 18, \quad 27, \quad 45, \quad 84, \quad 81,$$

respectively, and are irreducible and inequivalent to their duals.

Proof. For the classical groups the calculation is immediate in the defining module. A nonidentity inner automorphism of order three of $\mathfrak{su}(n)$ has a representative with eigenvalues among $c, c\zeta, c\zeta^2$, where $\zeta^3 = 1, \zeta \neq 1$; the representative need not have order three. If $s \geq 2$ eigenspaces are nonempty, its centralizer $S(\prod_{j=1}^s U(n_j))$ has center dimension $s - 1 > 0$. In $SO(n)$ and $Sp(n)$, an order-three inner automorphism admits an order-three representative, since their centers have order coprime to three. A nonidentity such representative has a unitary factor in its centralizer, again giving a center. Thus no proper classical fixed algebra is semisimple. For involutions, the semisimple classical centralizers have self-dual tensor-product quotients.

For the exceptional groups the remaining calculation is finite. Starting from each Cartan matrix, simple reflections generate the roots in simple-root coordinates. A homomorphism $Q \rightarrow \mathbb{Z}_m$, $m = 2, 3$, is determined by its values on the simple roots. We enumerate all $m^{\text{rank } \mathfrak{g}} - 1$ nonzero assignments and form their Weyl orbits. This covers every toral grading under consideration. We retain the gradings whose grade-zero roots span the full rank and test connectedness of the grade-one weight graph. The exact implementation and resulting root data are supplied in the computational companion [29]. Its vertices are the roots of degree one; two are joined when their difference is a grade-zero root. The corresponding root operator acts nontrivially between their one-dimensional weight spaces. Since the grade-zero algebra contains the full Cartan subalgebra, its submodules are sums of these weight spaces, and the connected components are precisely the irreducible constituents. For $m = 2$ every semisimple row has a connected self-conjugate odd module. For $m = 3$ the six rows

displayed above are the complete list. The Killing form pairs grades one and two nondegenerately, so the two modules are dual; the connected grade-one graphs prove irreducibility. $\square$

Lemma 3.4 (Degree-three invariant theory). *For a simple complex Lie algebra $\mathfrak{s}$,*

$$(\mathrm{Sym}^3 \mathfrak{s}^*)^{\mathfrak{s}} \neq 0 \iff \mathfrak{s} \text{ has type } A_n \text{ with } n \geq 2.$$

Thus only type A factors can carry a nonzero perturbative cubic coefficient; a particular representation of such a factor may still have coefficient zero.

Proof. By Chevalley's theorem, invariant polynomials on $\mathfrak{s}$ have basic degrees equal to the exponents plus one. Degree three occurs precisely in type A_n , $n \geq 2$. $\square$

3.2 The rank-deficient branch

No classification of maximal subalgebras is needed in this branch. The exceptional ambient algebras are excluded by a finite arithmetic sieve; the classical ambient algebras admit a uniform highest-weight argument.

Proposition 3.5 (Exceptional rank-deficient no-go). *Let $\mathfrak{g}$ be exceptional simple and let $\mathfrak{h} \subset \mathfrak{g}$ be semisimple with $\mathrm{rank} \mathfrak{h} < \mathrm{rank} \mathfrak{g}$. If $(\mathfrak{g}/\mathfrak{h})_{\mathbb{C}} = V \oplus V^*$ with V irreducible of complex type, then $\mathrm{Rad}(c_V) = 0$.*

Proof. Write $\mathfrak{h} = \bigoplus_i \mathfrak{h}_i$ and $V = \boxtimes_i V_i$. Let $d_i = \dim V_i$. Normalize the quadratic index by taking long roots to have squared length two and requiring $T(\mathrm{ad}_{\mathfrak{h}_i}) = h_i^{\vee}$; in particular $T_{A_{n-1}}(\mathbf{n}) = \frac{1}{2}$. If $I_i \in \mathbb{Z}_{>0}$ is the Dynkin index of $\mathfrak{h}_i \hookrightarrow \mathfrak{g}$, restriction of the adjoint representation gives the necessary equations

$$\dim \mathfrak{g} = \sum_i \dim \mathfrak{h}_i + 2 \prod_i d_i, \quad h_{\mathfrak{g}}^{\vee} I_i = h_i^{\vee} + 2T_i \prod_{j \neq i} d_j. \quad (6)$$

The normalisation is visible in the surviving full-rank row. For $E_8 \supset E_6 + A_2$, both embeddings have index one and

$$30 = 12 + 2 \cdot 3 \cdot 3, \quad 30 = 3 + 2 \cdot \frac{1}{2} \cdot 27. \quad (6a)$$

The radical is nonzero only if at least one simple factor has vanishing cubic coefficient on V_i ; Lemma 3.4 makes this condition automatic for every non- A factor.

The dimension equation bounds the representation search before any index is computed. Since $\mathfrak{h} \neq 0$ is semisimple, rank deficient, and $\dim \mathfrak{g} - \dim \mathfrak{h} = 2 \dim V$, the smallest parity-compatible possibility for $\dim \mathfrak{h}$ gives

$\mathfrak{g}$	G_2	F_4	E_6	E_7	E_8
smallest compatible $\dim \mathfrak{h}$	none	6	6	3	6
max $\dim V$	none	23	36	65	121.

(6b)

For G_2 , the only smaller-rank semisimple type is A_1 , and $14 - 3$ is odd. For the even-dimensional ambient algebras the smallest parity-compatible semisimple dimension is six, while E_7 admits A_1 of dimension three.

There are finitely many multisets of simple factor types of smaller total rank. For each factor, enumerate dominant Dynkin labels from zero by successively increasing one label. Weyl's dimension formula is strictly increasing in each label, so a branch may be stopped when its dimension exceeds the applicable bound in (6b). This terminates and includes every admissible highest weight. Each factor dimension is at most $\dim V$, so the same bounds suffice for every factor.

For every multiset, form all representation products within these bounds, identify permutations of identical factors, and impose the dimension equation, complex type, a zero cubic coefficient on at least one factor, and positive integrality of every index in (6). The finite calculation leaves no candidate in any of the five ambient types [29]. Its machine-readable audit records every product reaching the index test and the nonintegral index excluding it. These are necessary conditions on an embedding; their empty intersection is sufficient for the no-go conclusion. $\square$

Lemma 3.6 (Defining modules and canonical constituents). *Let $\mathfrak{g}_{\mathbb{C}}$ be $\mathfrak{sl}(W)$, $\mathfrak{so}(W)$, or $\mathfrak{sp}(W)$, and suppose $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ is semisimple and maximal with quotient $V \oplus V^*$, where $V \not\cong V^*$ is irreducible. Then W is irreducible as an $\mathfrak{h}_{\mathbb{C}}$ -module and $\mathfrak{h}_{\mathbb{C}}$ is simple.*

If λ is the highest weight of W , let λ^ be the highest weight of W^* , and write $i \mapsto i^*$ for the Dynkin-node involution induced by duality. The following constituents occur with multiplicity one:*

$$V_{\lambda+\lambda^*} \subset W \otimes W^*, \quad V_{2\lambda} \subset \text{Sym}^2 W,$$

and, when W is orthogonal,

$$V_{2\lambda-\alpha_i} \subset \Lambda^2 W \quad \text{for every } i \text{ with } \langle \lambda, \alpha_i^\vee \rangle > 0. \quad (6c)$$

If the last integer is at least three, then $V_{2\lambda-3\alpha_i}$ also occurs in $\Lambda^2 W$.

Proof. Irreducibility of W . Suppose first that the defining module is reducible. In the compact real form, complete reducibility gives either a proper invariant orthogonal decomposition or an irreducible real module whose complexification splits as an isotropic dual pair. In the first case let B be the compact block stabilizer. If $\mathfrak{h} \subsetneq B$, maximality is contradicted. If $\mathfrak{h} = B$, then in the unitary case B has nonzero center, contrary to semisimplicity; in the orthogonal and symplectic cases the off-diagonal quotient is a sum of self-dual tensor products of defining blocks, contrary to irreducible complex type. Under the isotropic-dual-pair alternative, the splitting defines an $\mathfrak{h}$ -invariant orthogonal complex structure, and $\mathfrak{h}$ lies in its proper compact unitary centralizer, whose nonzero centre excludes equality with the semisimple $\mathfrak{h}$, again contradicting maximality. Thus the defining module is irreducible.

Simplicity of $\mathfrak{h}$. Write $W = \boxtimes_i W_i$ for the simple ideals of $\mathfrak{h}$. In $\mathfrak{sl}(W)$, two distinct factors produce the self-dual constituent $\text{ad } \mathfrak{h}_i \boxtimes \text{ad } \mathfrak{h}_j$ outside $\mathfrak{h}$. In the orthogonal or symplectic cases every W_i is self-dual. Choose its invariant nondegenerate bilinear form, symmetric or alternating as appropriate. Adjointness in the decomposition $\text{End}(W_i) = K_i \oplus P_i$ refers to this bilinear form: K_i is the skew-adjoint part and P_i the self-adjoint part. With $P_i^0 = \{p \in P_i : \text{Tr } p = 0\}$ for the traceless self-adjoint part, the ambient Lie algebra is the odd-parity part of $\bigotimes_i (K_i \oplus P_i)$. Three factors again give the self-dual constituent $\text{ad } \mathfrak{h}_1 \boxtimes \text{ad } \mathfrak{h}_2 \boxtimes \text{ad } \mathfrak{h}_3$. With two factors, the nonzero blocks $\text{ad } \mathfrak{h}_1 \boxtimes P_2^0$ and $P_1^0 \boxtimes \text{ad } \mathfrak{h}_2$ lie in disjoint tensor summands outside $\mathfrak{h}$ and are self-dual by their nondegenerate trace pairings. But $V \oplus V^*$ has no nonzero proper self-dual submodule, so it cannot contain both blocks. If $P_1^0 = 0$, then W_1 is two-dimensional symplectic. When $P_2^0 \neq 0$, maximality forces $\mathfrak{h}_2 = K_2$, and the quotient is $\text{ad } \mathfrak{sl}_2 \boxtimes P_2^0$, an irreducible self-dual module; if both traceless parts vanish, the quotient is zero. Hence $\mathfrak{h}$ is simple.

Highest-weight constituents. The first two displayed inclusions are the Cartan products. For the third, if v_λ is a highest-weight vector and f_i a simple lowering operator, then $v_\lambda \wedge f_i v_\lambda$ is a highest-weight vector of weight $2\lambda - \alpha_i$. The corresponding weight space in $\Lambda^2 W$ is one-dimensional, which proves the stated multiplicity. If $m = \langle \lambda, \alpha_i^\vee \rangle \geq 3$, the vector

$$m v_\lambda \wedge f_i^3 v_\lambda - 3(m-2) f_i v_\lambda \wedge f_i^2 v_\lambda$$

is nonzero and highest of weight $2\lambda - 3\alpha_i$. In the remaining dual-node cases, this weight and its dual are distinct from the adjoint and from the pair $2\lambda - \alpha_i, 2\lambda - \alpha_{i^*}$; they therefore supply an additional quotient pair. $\square$

Proposition 3.7 (Classical rank-deficient no-go). *Let $(\mathfrak{g}, \mathfrak{h})$ satisfy the standing pair hypotheses, let $\mathfrak{g}$ be classical simple, and let $\mathfrak{h} \subset \mathfrak{g}$ be rank deficient. Under condition (1), the radical of the complex half is zero.*

Proof. Assume for contradiction that $\text{Rad}(c_V) \neq 0$. Then the standing pair hypotheses, condition (1), and Lemma 2.2 imply that $\mathfrak{h}$ is semisimple and maximal, so Lemma 3.6 applies; use its notation. If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(W)$, the self-dual Cartan constituent $V_{\lambda+\lambda^*}$ must be the adjoint module of $\mathfrak{h}$; otherwise it would be a self-dual summand of the quotient. Thus $\lambda + \lambda^* = \theta$, where θ is the highest root of $\mathfrak{h}_{\mathbb{C}}$. The three Cartan-product equations used in this proof are labelwise, and their nonzero dominant solutions are

$$\begin{aligned} \lambda + \lambda^* &= \theta & A_r : \omega_1, \omega_r; \quad C_r : \omega_1 \text{ (equivalently } B_2 : \omega_2), \\ 2\lambda &= \theta & C_r : \omega_1 \text{ (including } A_1, \text{ and equivalently } B_2 : \omega_2), \\ 2\lambda - \alpha_i &= \theta, \quad i = i^* & A_1 : 2\omega_1; A_3 : \omega_2; B_r : \omega_1; B_3 : \omega_3; C_2 : \omega_2; \\ & & D_r : \omega_1; D_4 : \omega_3, \omega_4; G_2 : \omega_1. \end{aligned} \tag{6d}$$

In the symplectic case, $\mathfrak{g}_{\mathbb{C}} \cong \text{Sym}^2 W$, and the self-dual Cartan constituent $V_{2\lambda}$ must likewise be the adjoint of $\mathfrak{h}$. This gives the second equation, $2\lambda = \theta$. The first row gives the standard A_r representation, for which the ambient algebra equals $\mathfrak{h}$, and the defining symplectic representation, whose quotient in $\mathfrak{sl}(W)$ is self-dual. The second row gives only the standard symplectic algebra and low-dimensional isomorphisms, so $\mathfrak{sp}(W)$ has no proper complex-type quotient.

Suppose $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(W)$. For every support node fixed by duality, the constituent in (6c) is self-dual and must equal the adjoint, so $2\lambda - \alpha_i = \theta$. The third row of (6d) consists of defining representations and low-rank isomorphisms, together with $G_2 \subset SO(7)$ and the spin embedding $B_3 \subset SO(8)$; their quotients are zero or self-dual. For distinct support nodes, the weights $2\lambda - \alpha_i$ are distinct. Once the self-dual cases have been excluded, each must be one of the two quotient highest weights. There can therefore be only one pair of support nodes exchanged by duality. Their coefficients are equal because the orthogonal module W is self-dual. Write this common coefficient as a . Such a pair occurs only in types A_r , odd D_r , and E_6 .

For A_{N-1} , choose the pair $i, N-i$ with $1 \leq i < N/2$. For a highest weight $\mu = \sum_{j=1}^{N-1} b_j \omega_j$, put $\ell_j = \sum_{k=j}^{N-1} b_k$, $\ell_N = 0$, and $x_j = \ell_j + N - j$. We use the cubic Harish–Chandra normalisation

$$C_3(\mu) = \sum_{j=1}^N (x_j - \bar{x})^3, \quad \bar{x} = \frac{1}{N} \sum_{j=1}^N x_j.$$

In this normalisation the constituent $\mu_i = 2\lambda - \alpha_i$ has

$$C_3(\mu_i) = -6a(2a + N - 2i) \neq 0. \tag{6e}$$

For completeness, let u, v be the i -th and $(i+1)$ -st centred coordinates of 2λ . Self-duality makes their full cubic sum zero, while

$$u - v = 2a + 1, \quad u + v = 2a + N - 2i.$$

Subtracting α_i changes only these two centred coordinates, to $u - 1, v + 1$. Hence the change in the cubic sum is

$$(u - 1)^3 - u^3 + (v + 1)^3 - v^3 = 3(u + v)(v - u + 1) = -6a(2a + N - 2i),$$

which proves (6e). In type A , the cubic trace tensor is a multiple of the unique invariant cubic. Contracting it with that invariant yields the cubic central eigenvalue times the dimension, up to a fixed nonzero normalisation. Thus its coefficient vanishes exactly when C_3 does. The constituent μ_i is consequently a quotient half with nonzero cubic index, so the radical is zero.

For odd D_r and for E_6 , a coefficient $a \geq 3$ gives the additional constituent $V_{2\lambda-3\alpha_i}$, contradicting the assumption that the quotient has only one dual pair. It remains to inspect $a = 1, 2$. Put

$$D_{r,a} = \dim V_{a(\omega_{r-1}+\omega_r)}, \quad M_{r,a} = \dim V_{2a(\omega_{r-1}+\omega_r)-\alpha_{r-1}}.$$

For odd $r \geq 5$ and $a \in \{1, 2\}$, we will show

$$\binom{D_{r,a}}{2} > r(2r-1) + 2M_{r,a}.$$

Thus the adjoint and the proposed dual pair cannot exhaust $\Lambda^2 W$. Weyl's formula gives the ratios

$$\frac{4M_{r,1}}{D_{r,1}^2} = \frac{36(r-1)(2r+1)}{r(r+2)(r+3)(r+4)},$$

$$\frac{4M_{r,2}}{D_{r,2}^2} = \frac{5,376,000(r-1)(r+2)(2r+1)(2r+3)^2(2r+5)}{r(r+1)(r+3)^2(r+4)^3(r+5)^3(r+6)^2(r+7)(r+8)}.$$

Set $R_{r,a} = 4M_{r,a}/D_{r,a}^2$. The two sequences decrease for $r \geq 5$. Indeed, after clearing the positive denominators in $R_{r+1,a}/R_{r,a} < 1$ and writing $n = r - 5$, the two differences are

$$4n^3 + 57n^2 + 259n + 365$$

and

$$32n^7 + 1,624n^6 + 34,692n^5 + 404,090n^4 \\ + 2,769,860n^3 + 11,162,622n^2 + 24,455,216n + 22,419,744,$$

respectively. Hence

$$R_{r,1} \leq R_{5,1} = \frac{22}{35} < \frac{2}{3}, \quad R_{r,2} \leq R_{5,2} = \frac{1274}{8019} < \frac{1}{6}.$$

Moreover $D_{r,1} = \binom{2r}{r-1} \geq r(2r-1)$, and monotonicity in the highest weight gives $D_{r,2} > D_{r,1}$. Writing $H = r(2r-1)$, we obtain

$$\binom{D_{r,1}}{2} - H - 2M_{r,1} \geq \frac{H(H-9)}{6} > 0, \\ \binom{D_{r,2}}{2} - H - 2M_{r,2} \geq \frac{H(5H-18)}{12} > 0.$$

Thus $\Lambda^2 W$ contains more than the adjoint and one dual pair. For E_6 , the two possible supports are $\{\omega_1, \omega_6\}$ and $\{\omega_3, \omega_5\}$; the four remaining values $a = 1, 2$ give the positive dimension gaps

$$52,897, \quad 3,008,865,288, \quad 1,949,250,433, \quad 24,307,954,884,506,865.$$

These four dimension gaps follow from Weyl's product formula; their exact values and a separate check of the symbolic inequalities above are recorded in the computational companion [29]. The inequalities are uniform in rank; the finite regression tests supplied with the code are supplementary checks, not the all-rank argument. No rank-deficient classical pair can have a nonzero radical. $\square$

Table 1: The direct full-rank census and its cubic coefficients. The radical is the sum of the factors with zero coefficient.

pair $\mathfrak{g} \supset \mathfrak{h}$	complex summand V	$\mathbf{A}(V)$	$\text{Rad}(c_V)$
$\mathfrak{g}_2 \supset \mathfrak{su}(3)$	$\mathbf{3}$	1	0
$\mathfrak{f}_4 \supset \mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathbf{6} \boxtimes \bar{\mathbf{3}}$	(21, -6)	0
$\mathfrak{e}_6 \supset \mathfrak{su}(3)^{\oplus 3}$	$\mathbf{3} \boxtimes \mathbf{3} \boxtimes \mathbf{3}$	(9, 9, 9)	0
$\mathfrak{e}_7 \supset \mathfrak{su}(6) \oplus \mathfrak{su}(3)$	$\mathbf{15} \boxtimes \bar{\mathbf{3}}$	(6, -15)	0
$\mathfrak{e}_8 \supset \mathfrak{su}(9)$	$\Lambda^3 \mathbf{9}$	9	0
$\mathfrak{e}_8 \supset \mathfrak{e}_6 \oplus \mathfrak{su}(3)_M$	$\mathbf{27} \boxtimes \mathbf{3}$	(0, 27)	$\mathfrak{e}_6$

3.3 Cubic indices and the survivor

For a simple factor of type A_{N-1} with $N \geq 3$, let $A(R)$ be the cubic index, normalised by $A(\mathbf{N}) = 1$; for A_1 and every other simple type put $A(R) = 0$. Duality gives $A(R^*) = -A(R)$, and for $V = \boxtimes_i V_i$ the coefficient on the i -th factor is

$$A_i(V) = A(V_i) \prod_{j \neq i} \dim V_j. \quad (6f)$$

Write $\mathbf{A}(V) := (A_i(V))_i$. Mixed cubics between distinct simple ideals vanish because they factor through the trace of a generator. Hence $\text{Rad}(c_V)$ is the sum of the simple ideals whose coefficients in (6f) vanish.

Table 1 gives the cubic coefficients of the six full-rank rows in Proposition 3.3.

The entries follow from the standard cubic-index formulas [14, 15]: for $N \geq 3$ and $1 \leq k \leq N-1$,

$$A(\Lambda^k \mathbf{N}) = \frac{(N-2k)(N-3)!}{(k-1)!(N-k-1)!}, \quad A(\text{Sym}^2 \mathbf{N}) = N+4,$$

and the tensor-product rule (6f); the trivial exterior powers at $k=0, N$ have index zero. The Lie algebra E_6 has no invariant symmetric cubic, whereas A_2 does. The order-three grading produces the pair $E_6 + A_2$; degree-three invariant theory distinguishes the anomalous factor from the radical. Thus the last row has

$$\mathbf{A}(V) = (0, 27), \quad \text{Rad}(c_V) = \mathfrak{e}_6.$$

The cubic on the $\mathbf{27}$ is a trilinear form on the representation; the vanishing used here concerns invariant trilinear forms on the Lie algebra $\mathfrak{e}_6$.

Proof of Theorem 2.1. Lemma 2.2 places every pair in the algebraic domain into the same structural normal form. Lemma 3.2 and Proposition 3.3 give the complete full-rank branch; Table 1 leaves only its final row. Propositions 3.5 and 3.7 exclude every rank-deficient pair with nonzero radical. The final row satisfies all hypotheses, proving the converse. $\square$

Remark (Completeness of the classification). The proof uses no classification of maximal subalgebras or homogeneous spaces. The full-rank branch is a finite toral-grading census from Cartan matrices; the exceptional rank-deficient branch is a finite Weyl-dimension/index sieve; and the classical rank-deficient branch is the uniform highest-weight argument above. The manuscript proves the reductions and establishes their completeness; the accompanying programs perform the remaining finite calculations.

3.4 The centralizer and multiplicity three

For the unique pair, the radical and its centralizer are

$$\text{Rad}(c_V) = \mathfrak{e}_6, \quad C_{\mathfrak{h}}(\text{Rad}(c_V)) = \mathfrak{su}(3)_M.$$

Here $C_{\mathfrak{h}}$ denotes the centralizer in $\mathfrak{h}$. The radical is the largest ideal on which all cubic coefficients, including mixed ones, vanish. Its centralizer acts on the second tensor factor, and its defining $\mathbf{3}$ gives three copies of the $\mathbf{27}$ after restriction to E_6 . The subscript M records this multiplicity. Section 5 computes the cubic on this factor. The adjoint branching displays V and V^* as the two halves of the adjoint complement:

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \overline{\mathbf{3}}). \quad (7)$$

Restriction to the first factor gives $V|_{E_6^{\text{sc}}} = 3\mathbf{27}$ and $V^*|_{E_6^{\text{sc}}} = 3\overline{\mathbf{27}}$. Either orientation therefore has multiplicity three.

3.5 The global exceptional subgroup

The same branching also determines the global image inside E_8 . The Lie-algebra inclusion integrates to a homomorphism $\varphi : E_6^{\text{sc}} \times SU(3)_M \rightarrow E_8$ with finite central kernel. Choose generators $z_E \in Z(E_6^{\text{sc}})$ and $z_3 \in Z(SU(3)_M)$ that act by the same primitive cube root ζ_3 on $\mathbf{27}$ and $\mathbf{3}$. The adjoint decomposition (7) shows that $\langle (z_E, z_3^{-1}) \rangle$ acts trivially on every summand. Conversely, a central kernel element lies in the product of the two order-three centers, and its action on $\mathbf{27} \boxtimes \mathbf{3}$ forces it into this diagonal subgroup. Faithfulness of the adjoint representation of E_8 leaves the kernel $\langle (z_E, z_3^{-1}) \rangle$. The global subgroup is

$$\frac{E_6^{\text{sc}} \times SU(3)_M}{\langle (z_E, z_3^{-1}) \rangle} \subset E_8.$$

The image of $(z_E, 1)$ acts on $\mathfrak{h}_{\mathbb{C}}, V, V^*$ with eigenvalues $1, \zeta_3, \zeta_3^{-1}$, respectively, and realizes (7) as a $\mathbb{Z}/3$ -grading. The branching is classical [14, 16]; the classification above shows that it is the only row satisfying (1) and (3).

4 Extremal intersections in E_6

The Lie-algebra classification leaves a single embedded E_6^{sc} , up to E_8 -conjugacy. Theorem 2.1 fixes the pair $E_6 + A_2 \subset E_8$ up to automorphisms of $\mathfrak{e}_8$, and all automorphisms of compact $\mathfrak{e}_8$ are inner.

We now work inside this E_6 . Closed root subsystems determine connected regular subgroups, and their intersections determine the semisimple roots of common connected subgroups. Choosing one maximal proper subsystem from each Weyl orbit gives a finite configuration space. We study two Weyl-invariant questions on it: simultaneous pairwise maximality and maximal common intersection.

The calculation rests on three pairwise facts. A $D_5 - (A_5 + A_1)$ intersection has two possible types; the $D_5 - A_2^3$ intersection always has type $A_2 + A_1 + A_1$; and an $(A_5 + A_1) - A_2^3$ intersection is large precisely when the distinguished root pair lies in the A_2^3 subsystem. The two extremal theorems combine these facts in different ways.

4.1 The finite configuration space

The three maximal subsystem types admit concrete parametrisations that make the finite problem explicit. Fix a maximal torus $T_0 \subset E_6^{\text{sc}}$ with Lie algebra $\mathfrak{t}_0$, let $\Phi = \Phi(E_6)$, and write $\mathcal{W} = W(E_6)$.

We use Bourbaki numbering: the chain is $1 - 3 - 4 - 5 - 6$, with node 2 attached to node 4. The Cartan matrix has diagonal entries 2, entries -1 on these edges, and zero elsewhere. A subsystem $\Psi \subseteq \Phi$ is *closed* when

$$\alpha, \beta \in \Psi, \quad \alpha + \beta \in \Phi \implies \alpha + \beta \in \Psi.$$

For such Ψ , let $\hat{\Psi}$ be the connected regular subgroup containing T_0 with

$$\text{Lie}(\hat{\Psi})_{\mathbb{C}} = (\mathfrak{t}_0)_{\mathbb{C}} \oplus \bigoplus_{\beta \in \Psi} (\mathfrak{e}_6)_{\beta}.$$

The intersections of these regular groups retain the common maximal torus T_0 . Passing to the derived subgroup retains only the torus generated by the common coroots; after the third intersection, some of these directions can become central. For two subsystems P, Q , define

$$H_{PQ} = [(\hat{P} \cap \hat{Q})^{\circ}, (\hat{P} \cap \hat{Q})^{\circ}].$$

Here the superscript $\circ$ denotes the identity component, and $[L, L]$ is the derived subgroup of L . The group H_{PQ} is the semisimple intermediate group obtained by intersecting two regular groups and taking the derived subgroup. We then intersect this intermediate group with the third regular group and take the identity component. This is the *pair-first construction*. The derived-subgroup step makes the choice of the first pair matter. We choose the pair whose intermediate group has greatest dimension; Proposition 4.4 proves that this choice is unique for every triple.

Proposition 4.1 (Maximal closed subsystems of E_6). *The maximal proper closed subsystems of Φ form exactly three $\mathcal{W}$ -orbits*

$$\mathcal{O}_D (D_5), \quad \mathcal{O}_A (A_5 + A_1), \quad \mathcal{O}_{\Theta} (A_2^3).$$

Let ω_1 be a minuscule fundamental weight, whose Weyl orbit has 27 elements, and put $\Lambda_{27} = \mathcal{W}\omega_1$, and let $\mathcal{R} = \Phi / \{\pm 1\}$. Here $\beta^{\vee}$ denotes the coroot of β . Then

$$\begin{aligned} D_{\lambda} &= \{\beta \in \Phi : \langle \lambda, \beta^{\vee} \rangle = 0\}, \quad \lambda \in \Lambda_{27}, \\ A_{\bar{\alpha}} &= \{\pm \alpha\} \cup (\Phi \cap \alpha^{\perp}), \quad \bar{\alpha} = \{\pm \alpha\} \in \mathcal{R}, \end{aligned}$$

give $\mathcal{W}$ -equivariant bijections $\Lambda_{27} \simeq \mathcal{O}_D$ and $\mathcal{R} \simeq \mathcal{O}_A$.

Proof. The Cartan matrix generates all 72 roots by simple reflections. Fix the resulting positive system. Enumerate all linearly independent subsets of positive roots with pairwise nonpositive inner products, including the empty subset; take their reflection closures, test additive closedness, and identify duplicates. This enumeration is exhaustive: every closed subsystem inherits a positive system from Φ , and its simple roots are one of the enumerated subsets. Reflection closure of those simple roots recovers the subsystem.

The exact traversal gives 5,079 closed subsystems, including the empty and full systems, and 103 maximal proper subsystems. Grouping the latter under the simple reflections gives 27 of type D_5 , 36 of type $A_5 + A_1$, and 40 of type A_2^3 [29]. A second implementation in $\mathbb{R}^8$ reproduces these counts by a different traversal: starting with the empty system, it adjoins opposite root pairs and takes additive closure. Every closed subsystem is reached, since adjoining its own root pairs keeps each intermediate closure inside it. Component ranks and Dynkin diagrams are recovered directly from the coordinates in this second census.

Roots orthogonal to a minuscule weight form D_5 ; a root pair $\{\pm \alpha\}$ determines the A_1 that it spans together with the orthogonal A_5 . If $D_{\lambda} = D_{\mu}$, their one-dimensional orthogonal complements give $\mu = \pm \lambda$; the negative weights lie in the dual minuscule orbit, so $\mu = \lambda$. Likewise, $A_{\bar{\alpha}}$ recovers $\{\pm \alpha\}$ as its A_1 -component. The orbit classification and these injectivity statements give both bijections. $\square$

Remark (The mark-three subsystem). The orbit $\mathcal{O}_\Theta$ is the Weyl orbit of the fixed-root subsystem of the mark-three toral grading of E_6 in Proposition 3.3. The two inverse nontrivial characters interchange the grade-one and grade-two modules, identifying them with the mutually dual 27-dimensional A_2^3 -modules displayed in the adjoint branching below. They are modules for the grade-zero algebra A_2^3 , not the fundamental **27** and $\overline{\mathbf{27}}$ of E_6 .

4.2 The two extremal orbits

The first criterion maximizes every pairwise intersection; the second maximizes the threefold intersection itself. Put

$$\mathcal{X} = \mathcal{O}_D \times \mathcal{O}_A \times \mathcal{O}_\Theta.$$

For subsystems P, Q drawn from distinct orbit factors, call $P \cap Q$ maximal when $|P \cap Q|$ attains its maximum over the full product of those two orbits. Because

$$\dim_{\mathbb{R}}((\widehat{P} \cap \widehat{Q})^\circ) = 6 + |P \cap Q|,$$

maximizing $|P \cap Q|$ is equivalent to maximizing the dimension of the identity component. Let $\mathcal{X}_{\max}$ be the set of triples for which all three pairwise intersections are largest.

Theorem 4.2 (Triples with maximal pairwise intersections). *The pairwise intersections are*

<i>pair</i>	<i>root-system type</i>	$ P \cap Q $
$D_\lambda \cap A_{\bar{\alpha}}$	$A_4, \quad \langle \lambda, \alpha^\vee \rangle \neq 0,$	20
	$A_3 + A_1 + A_1, \quad \langle \lambda, \alpha^\vee \rangle = 0$	16
$D_\lambda \cap \Theta$	$A_2 + A_1 + A_1$	10
$A_{\bar{\alpha}} \cap \Theta$	$A_2 + A_2 + A_1, \quad \{\pm\alpha\} \subset \Theta,$	14
	$A_1 + A_1 + A_1, \quad \{\pm\alpha\} \not\subset \Theta$	6

and therefore

$$\mathcal{X}_{\max} = \{(D_\lambda, A_{\bar{\alpha}}, \Theta) : \langle \lambda, \alpha^\vee \rangle \neq 0, \{\pm\alpha\} \subset \Theta\}.$$

For every $s = (D_\lambda, A_{\bar{\alpha}}, \Theta) \in \mathcal{X}_{\max}$,

$$D_\lambda \cap A_{\bar{\alpha}} \cong A_4, \quad R_s := D_\lambda \cap A_{\bar{\alpha}} \cap \Theta \cong A_2 + A_1.$$

Its stabilizer and orbit size are

$$\text{Stab}_{\mathcal{W}}(s) = W(R_s), \quad \mathcal{X}_{\max} \simeq \mathcal{W}/W(R_s), \quad |\mathcal{X}_{\max}| = 4320.$$

Thus $\mathcal{X}_{\max}$ is a single $\mathcal{W}$ -orbit.

Proof. Pairwise types. The required branchings [14, 15] are

$$\begin{aligned} \mathbf{27}|_{A_5+A_1} &= (\mathbf{15}, \mathbf{1}) \oplus (\overline{\mathbf{6}}, \mathbf{2}), \\ \mathbf{27}|_{A_2^3} &= (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}) \oplus (\overline{\mathbf{3}}, \mathbf{1}, \mathbf{3}), \\ \mathfrak{e}_6|_{A_2^3} &= (\mathbf{8}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{8}) \\ &\quad \oplus (\mathbf{3}, \mathbf{3}, \mathbf{3}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{3}}, \overline{\mathbf{3}}). \end{aligned}$$

In the first branching, a weight in $(\mathbf{15}, \mathbf{1})$ has orthogonal roots $A_3 + A_1$ inside A_5 , together with the external A_1 ; this gives $A_3 + A_1 + A_1$. A weight in $(\overline{\mathbf{6}}, \mathbf{2})$ instead has orthogonal roots A_4 . These

are respectively the cases $\langle \lambda, \alpha^\vee \rangle = 0$ and $\langle \lambda, \alpha^\vee \rangle \neq 0$. In the second branching, every weight is nontrivial on two A_2 -factors and trivial on the third. A weight of $\mathbf{3}$ or $\bar{\mathbf{3}}$ is orthogonal to one root pair in A_2 , so $D_\lambda \cap \Theta \cong A_2 + A_1 + A_1$.

The common root system. The adjoint branching shows that $A_{\bar{\alpha}} \cap \Theta$ contains two full A_2 -factors and the A_1 generated by α when $\{\pm\alpha\} \subset \Theta$. Otherwise α is a weight of $(\mathbf{3}, \mathbf{3}, \mathbf{3})$ or its dual and is orthogonal to one root pair in each A_2 -factor, giving $A_1 + A_1 + A_1$. The root counts in the table follow. Their strict inequalities give the displayed description of $\mathcal{X}_{\max}$. If α lies in one A_2 -factor, a λ satisfying $\langle \lambda, \alpha^\vee \rangle \neq 0$ is nontrivial on that factor and exactly one other factor. The third factor contributes A_2 , while the other nontrivial factor contributes A_1 . Hence $R_s \cong A_2 + A_1$.

The stabilizer. For $s \in \mathcal{X}_{\max}$, Proposition 4.1 shows that a stabilizer of D_λ fixes λ . A stabilizer of $A_{\bar{\alpha}}$ sends α to $\pm\alpha$, and the nonzero pairing excludes the minus sign. The pair stabilizer therefore fixes $L = \text{span}_{\mathbb{R}}\{\lambda, \alpha\}$ pointwise. Choose $v \in L$ outside the fixed subspaces of Weyl elements that do not fix L pointwise. Then $\text{Stab}_{\mathcal{W}}(v)$ is the pair stabilizer, $\Phi \cap v^\perp = D_\lambda \cap A_{\bar{\alpha}}$, and the Weyl point-stabilizer theorem [13, Ch. V] gives

$$\text{Stab}_{\mathcal{W}}(D_\lambda, A_{\bar{\alpha}}) = W(D_\lambda \cap A_{\bar{\alpha}}) \cong W(A_4).$$

For every $\beta \in R_s$, one has $\beta \perp \lambda$; moreover, either $\beta = \pm\alpha$ or $\beta \perp \alpha$, and $\beta \in \Theta$. The reflection s_β preserves each of D_λ , $A_{\bar{\alpha}}$, and Θ , so $W(R_s)$ fixes the triple. Conversely, every triple stabilizer lies in the pair stabilizer and normalizes R_s ; therefore

$$W(R_s) \subseteq \text{Stab}_{\mathcal{W}}(s) \subseteq N_{W(A_4)}(R_s).$$

Inside A_4 , the subsystem $A_2 + A_1$ is the $3 + 2$ block decomposition. Its unequal block sizes give $N_{W(A_4)}(R_s) = W(R_s)$, of order 12. Since $|W(E_6)| = 51840$, the orbit of s has 4320 elements.

The orbit count. Affine-diagram symmetry [13, Ch. VI] gives

$$W(A_2)^3 \rtimes S_3 \subseteq N_{\mathcal{W}}(\Theta), \quad |\mathcal{O}_\Theta| \leq \frac{51840}{6^3 \cdot 6} = 40.$$

Each Θ contains nine root pairs. For a fixed root pair in one A_2 -factor, two weights of $\mathbf{3}$ or $\bar{\mathbf{3}}$ pair nontrivially with it; the other nontrivial factor contributes three choices in each of two summands. There are therefore $2 \cdot 3 + 2 \cdot 3 = 12$ compatible weights λ , and $|\mathcal{X}_{\max}| \leq 40 \cdot 9 \cdot 12 = 4320$. The orbit already found attains the bound, proving that the maximizers form one orbit and establishing the stabilizer statement. $\square$

For $\Theta \in \mathcal{O}_\Theta$, write $\Theta = C_1 + C_2 + C_3$ for its three irreducible A_2 -components. If $\{\pm\alpha\} \subset \Theta$, let C_α be the component containing α . We say that λ is nontrivial on C_α when $\langle \lambda, \beta^\vee \rangle \neq 0$ for some $\beta \in C_\alpha$.

Theorem 4.3 (Triples with largest common intersection). *The largest possible common intersection has ten roots. The maximizing set is*

$$\mathcal{X}_\cap = \left\{ (D_\lambda, A_{\bar{\alpha}}, \Theta) : \begin{array}{l} \{\pm\alpha\} \subset \Theta, \\ \langle \lambda, \alpha^\vee \rangle = 0, \\ \lambda \text{ is nontrivial on } C_\alpha \end{array} \right\}.$$

For every $s \in \mathcal{X}_\cap$,

$$R_s = D_\lambda \cap A_{\bar{\alpha}} \cap \Theta \cong A_2 + A_1 + A_1.$$

Its stabilizer and orbit size are

$$\text{Stab}_{\mathcal{W}}(s) = W(R_s), \quad \mathcal{X}_\cap \simeq \mathcal{W}/W(R_s), \quad |\mathcal{X}_\cap| = 2160.$$

The maximizing triples form a single $\mathcal{W}$ -orbit.

Proof. The maximal root count. Suppose first that $\{\pm\alpha\} \not\subset \Theta$. Then $A_{\bar{\alpha}} \cap \Theta \cong A_1^3$, so the common intersection has at most six roots.

Now let $\alpha \in C_1$. The restriction of the **27** to A_2^3 shows that λ is nontrivial on exactly two of the three factors and trivial on the third. If $\langle \lambda, \alpha^\vee \rangle \neq 0$, the factor containing α contributes nothing to the common intersection; the other two factors contribute $A_2 + A_1$, giving eight roots. If the pairing is zero and λ is trivial on C_1 , each factor contributes one root pair, giving A_1^3 and six roots. In the remaining case, λ is nontrivial on C_1 but orthogonal to α . The factor on which λ is trivial contributes a full A_2 ; the other nontrivial factor contributes an A_1 ; and $\{\pm\alpha\}$ contributes a second A_1 . Consequently,

$$R_s \cong A_2 + A_1 + A_1, \quad |R_s| = 10.$$

These cases prove both the characterization and the maximal value.

The count. Each Θ contains nine root pairs. For a fixed pair in one A_2 -factor, consider the two summands of $\mathbf{27}|_{A_2^3}$ that are nontrivial on that factor. In each summand, one of the three weights on the chosen factor is orthogonal to α , while the other nontrivial factor has three choices. This gives $3 + 3 = 6$ weights, and

$$|\mathcal{X}_\cap| = 40 \cdot 9 \cdot 6 = 2160.$$

The stabilizer. By Proposition 4.1, an element stabilizing D_λ fixes λ ; it cannot send λ to $-\lambda$, which lies in the dual minuscule orbit. A stabilizer of $A_{\bar{\alpha}}$ sends α to $\pm\alpha$. Put

$$H_0 = \{w \in \mathcal{W} : w\lambda = \lambda, w\alpha = \alpha\}.$$

Choose $v \in \text{span}_{\mathbb{R}}\{\lambda, \alpha\}$ outside the fixed subspaces of Weyl elements that do not fix both λ and α . Then $\text{Stab}_{\mathcal{W}}(v) = H_0$, and its orthogonal root system is $A_3 + A_1$. The Weyl point-stabilizer theorem therefore gives $H_0 = W(A_3 + A_1)$. Since $\langle \lambda, \alpha^\vee \rangle = 0$, the reflection s_α fixes λ , commutes with H_0 , and exchanges α with $-\alpha$. Therefore

$$\text{Stab}_{\mathcal{W}}(D_\lambda, A_{\bar{\alpha}}) = H_0 \times \langle s_\alpha \rangle = W(A_3 + A_1 + A_1).$$

For every $\beta \in R_s$, one has $\beta \perp \lambda$; moreover, either $\beta = \pm\alpha$ or $\beta \perp \alpha$, and $\beta \in \Theta$. Thus s_β preserves each member of the triple, so $W(R_s) \subseteq \text{Stab}_{\mathcal{W}}(s)$. Conversely, a triple stabilizer lies in the pair stabilizer and normalizes R_s . Inside $W(A_3 + A_1 + A_1)$, the subsystem R_s contains both A_1 -components and an A_2 in the A_3 -factor. In $W(A_3) \cong S_4$, the corresponding $S_3 = W(A_2)$ is self-normalizing, so

$$W(R_s) \subseteq \text{Stab}_{\mathcal{W}}(s) \subseteq N_{W(A_3+A_1+A_1)}(R_s) = W(R_s).$$

The stabilizer has order 24, so its orbit has $51840/24 = 2160$ elements. This is the number of maximizers, and the maximizing set is one orbit. $\square$

Since $|D_\lambda \cap \Theta| = 10$ for every pair (D_λ, Θ) , the common-maximum condition has the equivalent form

$$|D_\lambda \cap A_{\bar{\alpha}} \cap \Theta| = 10 \iff D_\lambda \cap \Theta \subseteq A_{\bar{\alpha}}.$$

The common maximum occurs when the third subsystem contains the rigid ten-root intersection of the other two.

The two extremal criteria are inequivalent. Pairwise maximality gives a common intersection of type $A_2 + A_1$ with eight roots, whereas maximizing the common intersection gives type $A_2 + A_1 + A_1$ with ten roots.

4.3 The largest intermediate group and the six triple orbits

We now compare the three intermediate groups H_{DA} , $H_{D\Theta}$, and $H_{A\Theta}$. Their dimensions give an intrinsic ordering before any matter representation is restricted.

Proposition 4.4 (The unique largest intermediate group). *For every $s = (D, A, \Theta) \in \mathcal{X}$, the three pairwise derived groups have the following root types and real dimensions:*

<i>intermediate group</i>	<i>root type</i>	$\dim H_{PQ}$
H_{DA}	A_4 or $A_3 + A_1 + A_1$	24 or 21
$H_{D\Theta}$	$A_2 + A_1 + A_1$	14
$H_{A\Theta}$	$A_2 + A_2 + A_1$ or A_1^3	19 or 9.

Thus H_{DA} is the unique group of greatest dimension. Its pair also has strictly more common roots than either of the other two pairs. The choice commutes with root-system automorphisms and, at group level, with conjugation of the entire configuration.

Proof. The pairwise types in Theorem 4.2 hold on all of $\mathcal{X}$. A compact semisimple group with root system Ψ has real dimension $\text{rank } \Psi + |\Psi|$. Hence the three rows have dimensions $20 + 4$ or $16 + 5$, $10 + 4$, and $14 + 5$ or $6 + 3$, respectively. The least possible value in the first row is 21, greater than every value in the other rows. Likewise, $16 > 14$ and $16 > 10$ prove the root-count assertion. Automorphisms preserve intersections and their ranks; conjugation preserves connected intersections, derived groups and dimension. $\square$

Having selected the intermediate group by its dimension, we now compute its chiral content and compare it with the other two orderings. To measure the remaining chirality, cancel each irreducible representation against its dual and discard self-dual summands.

For a representation $R = \bigoplus_U m_U U$ of a semisimple compact group, define its chiral dimension by

$$d_\chi(R) = \sum_{\{U, U^*\}} |m_U - m_{U^*}| \dim U,$$

where the sum runs over unordered pairs of non-self-dual irreducibles. Thus d_χ counts the dimension of the non-self-dual remainder. We evaluate it at each semisimple intermediate group, before further restriction. Section 5 identifies the resulting representations and charges.

Proposition 4.5 (Chiral dimensions of the three intermediate groups). *For distinct $P, Q \in \{D_\lambda, A_{\bar{\alpha}}, \Theta\}$, the restriction of the **27** to H_{PQ} has*

<i>intermediate group</i>	$d_\chi(\mathbf{27})$
H_{DA}	15 or 16
$H_{D\Theta}$	12
$H_{A\Theta}$	27 or 0.

Proof. For the two D - A strata, the familiar restrictions are

$$\mathbf{27}|_{A_4} = \mathbf{10} \oplus 2\mathbf{\bar{5}} \oplus \mathbf{5} \oplus \mathbf{21}$$

and

$$\mathbf{27}|_{A_3+A_1+A_1} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{\bar{4}}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{6}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}),$$

which give chiral dimensions 15 and 16.

For the other two orderings, use

$$\mathbf{27}|_{A_2^3} = (\mathbf{3}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{3}}, \mathbf{1}, \mathbf{3}).$$

The subgroup $H_{D\Theta}$ contains one full A_2 factor and one A_1 in each of the other two. In the order $A_1 \times A_1 \times A_2$,

$$\begin{aligned} \mathbf{27} = & (\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}) \\ & \oplus (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}). \end{aligned}$$

The first line is self-dual, and the two three-dimensional terms in the second line form a dual pair. The remaining two summands have dimension six, so $d_\chi = 12$.

If $H_{A\Theta}$ has type $A_1 + A_2 + A_2$, then

$$\mathbf{27} = (\mathbf{2}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}),$$

and none of the dual summands occurs, so $d_\chi = 27$. If $H_{A\Theta}$ has type A_1^3 , every irreducible is self-dual and $d_\chi = 0$. $\square$

Corollary 4.6 (Chirality of the largest intermediate group). *Among the three pair-first intermediate groups, the following are equivalent: the group has greatest dimension; its pair has the greatest number of common roots; and its restricted $\mathbf{27}$ has chiral dimension 15 or 16. The unique group satisfying these conditions is H_{DA} .*

Proof. Combine Propositions 4.4 and 4.5. $\square$

The values 15 and 16 are therefore consequences of the structural ordering. They are the chiral dimensions of the familiar $SU(5)$ and Pati–Salam one-family representations. On restriction to the Standard Model, the latter includes the neutral singlet ν^c , which no longer contributes to d_χ .

The next result identifies the triples whose common root system contains an A_2 component. Along the faithful chains below, this component is the colour factor.

Proposition 4.7 (The A_2 -containing orbits). *For $s = (D_\lambda, A_{\bar{\alpha}}, \Theta) \in \mathcal{X}$, put*

$$R_s = D_\lambda \cap A_{\bar{\alpha}} \cap \Theta.$$

Then R_s contains an irreducible component of type A_2 if and only if s is pairwise maximal or has largest common intersection.

More generally, define

$$S_s = [(\hat{D}_\lambda \cap \hat{A}_{\bar{\alpha}})^\circ, (\hat{D}_\lambda \cap \hat{A}_{\bar{\alpha}})^\circ], \quad K_s = (S_s \cap \hat{\Theta})^\circ.$$

The semisimple root system of K_s is R_s . Among all 38,880 triples, the extracted group K_s contains a simple $\mathfrak{su}(3)$ factor only on the two extremal classes.

Proof. Write $\Theta = C_1 + C_2 + C_3$. If $\{\pm\alpha\} \not\subset \Theta$, then $A_{\bar{\alpha}} \cap \Theta \cong A_1^3$, so every component of R_s has type A_1 . Now suppose $\alpha \in C_1$. The restriction of the $\mathbf{27}$ to A_2^3 shows that λ is nontrivial on exactly two factors. If $\langle \lambda, \alpha^\vee \rangle \neq 0$, the proof of Theorem 4.2 gives $R_s \cong A_2 + A_1$, and the triple is pairwise maximal. If the pairing vanishes and λ is nontrivial on C_1 , the proof of Theorem 4.3 gives $R_s \cong A_2 + A_1 + A_1$, and the common intersection is maximal. In the remaining case λ is trivial on C_1 , and $R_s \cong A_1^3$. An A_2 -component occurs only in the two extremal cases.

The roots of S_s are $D_\lambda \cap A_{\bar{\alpha}}$. Since $\hat{\Theta}$ contains the fixed maximal torus, the roots of the derived algebra of K_s are exactly their intersection with Θ , namely R_s , which proves the group statement. $\square$

Since $|\mathcal{O}_D| = 27$, $|\mathcal{O}_A| = 36$, and $|\mathcal{O}_\Theta| = 40$,

$$|\mathcal{X}| = 27 \cdot 36 \cdot 40 = 38,880.$$

For a triple $s = (D, A, \Theta)$, its pairwise profile is $p(s) = (|D \cap A|, |D \cap \Theta|, |A \cap \Theta|)$. To obtain the full orbit decomposition, enumerate every triple and form its orbit by simultaneously applying the six simple reflections to its three subsystems. Repeat from an unvisited triple until none remains. Because these reflections generate $\mathcal{W}$, the process gives the complete orbit partition, not merely a partition by numerical invariants. The resulting six orbits have the profiles, common root systems, and cardinalities in Table 2 [29]. Their sizes sum to 38,880; exactly one orbit realizes each displayed row.

Table 2: The six Weyl orbits in the triple space.

pairwise profile	$ R_s $	type of R_s	status	triples
(20, 10, 14)	8	$A_2 + A_1$	pairwise maximum	4,320
(16, 10, 14)	10	$A_2 + A_1 + A_1$	common maximum	2,160
(20, 10, 6)	4	$A_1 + A_1$	nonextremal	12,960
(16, 10, 14)	6	$A_1 + A_1 + A_1$	nonextremal	3,240
(16, 10, 6)	6	$A_1 + A_1 + A_1$	nonextremal	3,240
(16, 10, 6)	2	A_1	nonextremal	12,960

Remark (Full root-system automorphisms). Let τ be the nontrivial diagram automorphism of E_6 , and let w_0 be the longest Weyl element. Since $w_0\tau = -1$ on the root system and every closed root subsystem is stable under negation, τ acts on closed subsystems exactly as w_0^{-1} . Hence the six $W(E_6)$ -orbits above are already the six orbits under $\text{Aut } \Phi(E_6)$; no cells merge when the full root-system automorphism group is used, and each stabilizer doubles in order. On representations the outer class exchanges **27** and **$\overline{27}$** , but that orientation exchange leaves the unoriented subsystem classification unchanged.

4.4 Faithful subgroup chains

The common root systems record the derived semisimple parts of the connected intersections, not their full tori. The abelian directions are recovered from the ambient maximal torus after the indicated commutator step. In the pairwise-maximal case the root system $A_2 + A_1$, together with one surviving circle, gives the rank-four group $S(U(3) \times U(2))$; the faithful quotient is determined below.

The root intersections determine the local semisimple types. The next result identifies their faithful compact global forms and compares the two chains.

Remark (Fixed groups and the intersection construction). The regular groups associated with D_5 , $A_5 + A_1$, and A_2^3 are the connected fixed groups of two toral involutions and of the order-three grading. Their root systems are obtained by the finite calculations in Propositions 3.3 and 4.1. The defining toral elements, and hence their inner automorphisms, may be chosen in the same maximal torus and therefore commute. Their fixed groups need not commute; the construction uses their intersections. Intersections such as $G_{\text{SM}} = SU(5) \cap G_{\text{PS}}$ inside $\text{Spin}(10)$ are familiar from unified and orbifold constructions [17]. Theorems 4.2 and 4.3 add a classification statement: the two chains arise from the two extremal Weyl orbits in the full triple space, with their faithful global forms and compatible $\mathbb{Z}_6$ kernels.

Corollary 4.8 (The Standard Model and left-right subgroup chains). *For a pairwise maximizer $s_P = (D, A_P, \Theta)$, define*

$$L = [\hat{D}, \hat{D}], \quad S_P = [(\hat{D} \cap \hat{A}_P)^\circ, (\hat{D} \cap \hat{A}_P)^\circ], \quad K_P = (S_P \cap \hat{\Theta})^\circ.$$

Then

$$K_P \cong G_{\text{SM}} = S(U(3) \times U(2)) \subset S_P \cong SU(5) \subset L \cong \text{Spin}(10).$$

Its global form is

$$G_{\text{SM}} \cong \frac{SU(3) \times SU(2) \times U(1)_Y}{\mathbb{Z}_6}.$$

For a triple $s_T = (D, A_T, \Theta)$ attaining the common maximum, define

$$S_T = [(\hat{D} \cap \hat{A}_T)^\circ, (\hat{D} \cap \hat{A}_T)^\circ], \quad K_T = (S_T \cap \hat{\Theta})^\circ.$$

Then

$$K_T \cong G_{\text{LR}} \subset S_T \cong G_{\text{PS}} \subset \text{Spin}(10),$$

where the faithful Pati–Salam and left-right groups are

$$G_{\text{PS}} \cong \frac{SU(4) \times SU(2)_L \times SU(2)_R}{\langle (-I_4, -I_2, -I_2) \rangle}, \quad G_{\text{LR}} \cong \frac{SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}}{\mathbb{Z}_6}.$$

If s_P and s_T have the same D and Θ , then

$$G_{\text{SM}} \subset G_{\text{LR}} \subset G_{\text{PS}} \subset \text{Spin}(10) \subset E_6^{\text{sc}}, \quad Y = T_R^3 + \frac{B-L}{2}.$$

There are 8,640 such ordered pairs (s_P, s_T) .

Proof. The pairwise-maximal chain. For s_P , the root systems of L, S_P, K_P are $D_5, A_4, A_2 + A_1$. Pullback of the **27** to $\text{Spin}(10)$ contains the faithful half-spinor **16**; pullback to $SU(5)$ contains a faithful **5** or $\bar{\mathbf{5}}$. Hence $L \cong \text{Spin}(10)$ and $S_P \cong SU(5)$. The connected full-rank subgroup of $SU(5)$ with roots $A_2 + A_1$ is $S(U(3) \times U(2))$, with covering map

$$(g_3, g_2, z) \mapsto \text{diag}(z^{-2}g_3, z^3g_2)$$

and kernel $\mathbb{Z}_6$.

The common-maximal chain. For s_T , the root systems of S_T and K_T are $A_3 + A_1 + A_1$ and $A_2 + A_1 + A_1$. Under the universal cover $SU(4) \times SU(2)_L \times SU(2)_R$, the **27** contains

$$(\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{6}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}).$$

The common kernel on these summands is generated by $(-I_4, -I_2, -I_2)$, so its faithful image in $\text{Spin}(10)$ is

$$S_T \cong G_{\text{PS}} \cong \frac{SU(4) \times SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}.$$

Inside $SU(4)$, the full-rank subgroup with roots A_2 is $S(U(3) \times U(1))$. Accordingly, K_T is the image of

$$\begin{aligned} SU(3) \times SU(2)_L \times SU(2)_R \times U(1) &\longrightarrow S_T, \\ (g_3, g_L, g_R, z) &\longmapsto [\text{diag}(zg_3, z^{-3}), g_L, g_R]. \end{aligned}$$

Its kernel is cyclic of order six, generated by

$$\left(e^{2\pi i/3}I_3, -I_2, -I_2, e^{\pi i/3}\right).$$

The $U(1)$ generator in $SU(4)$ is proportional to $\text{diag}(1, 1, 1, -3)$, which is the usual $B - L$ generator, giving the stated global form of G_{LR} [18, 14, 15].

Containment of the root systems. Fix (D_λ, Θ) , and write $\Theta = C_1 + C_2 + C_3$ so that λ is nontrivial on C_1, C_2 and trivial on C_3 . For $i = 1, 2$, let $\{\pm\rho_i\}$ be the unique root pair in C_i orthogonal to λ . A pairwise maximizer chooses one of the other two root pairs in C_1 or C_2 , giving four choices for A_P . Its common root system is

$$C_3 + \{\pm\rho_2\} \quad \text{or} \quad C_3 + \{\pm\rho_1\},$$

according to the factor containing the chosen root. A triple attaining the common maximum chooses $\{\pm\rho_1\}$ or $\{\pm\rho_2\}$, giving two choices for A_T , and in either case its common root system is

$$C_3 + \{\pm\rho_1\} + \{\pm\rho_2\}.$$

For fixed (D_λ, Θ) , the common root subsystem is therefore independent of the two admissible choices of A_T . The root system of K_P is contained in that of K_T for all eight pairings.

The Cartan subalgebras are compatible as well. The roots of S_P span a four-dimensional subspace of $\lambda^\perp$, while the roots of S_T have rank five and span $\lambda^\perp$. Since $\widehat{\Theta}$ contains T_0 , the Cartan subalgebras of K_P and K_T are those of S_P and S_T , respectively, so $K_P \subset K_T$.

Compatibility of the global forms. Put $\omega = e^{2\pi i/3}$ and $\zeta = e^{\pi i/3}$. The covering kernels of G_{SM} and G_{LR} are generated, respectively, by

$$k_{\text{SM}} = (\omega I_3, -I_2, \zeta), \quad k_{\text{LR}} = (\omega I_3, -I_2, -I_2, \zeta).$$

With consistent orientations of the two abelian factors, the inclusion lifts to

$$\begin{aligned} \tilde{\iota} : SU(3) \times SU(2)_L \times U(1) &\longrightarrow SU(3) \times SU(2)_L \times SU(2)_R \times U(1), \\ (g_3, g_L, u) &\longmapsto \left(g_3, g_L, \begin{pmatrix} u^3 & 0 \\ 0 & u^{-3} \end{pmatrix}, u\right). \end{aligned}$$

It sends k_{SM} to k_{LR} , so it descends to $G_{\text{SM}} \hookrightarrow G_{\text{LR}}$. On Lie algebras its abelian generator is

$$6Y = 6T_R^3 + 3(B - L), \quad \text{equivalently} \quad Y = T_R^3 + \frac{B - L}{2}.$$

The containment count. There are $27 \cdot 40 = 1080$ choices of (D, Θ) . Each has four pairwise-maximal and two common-maximal choices of A , so there are $1080 \cdot 4 \cdot 2 = 8640$ ordered pairs (s_P, s_T) . $\square$

Remark (Covering groups and faithful quotients). Writing tildes for the displayed covering products, the covering-product and quotient maps form the commutative diagram

$$\begin{array}{ccccc} \tilde{G}_{\text{SM}} & \hookrightarrow & \tilde{G}_{\text{LR}} & \xrightarrow{\pi} & \tilde{G}_{\text{PS}} \\ \downarrow & & \downarrow & & \downarrow \\ G_{\text{SM}} & \hookrightarrow & G_{\text{LR}} & \hookrightarrow & G_{\text{PS}}, \end{array}$$

where the three vertical kernels are $\mathbb{Z}_6$, $\mathbb{Z}_6$, and $\mathbb{Z}_2$, respectively. The first upper arrow is the injective lift $\tilde{\iota}$ displayed above. The second is the natural covering-product map and has

$$\ker \pi = \langle k_{\text{LR}}^2 \rangle \cong \mathbb{Z}_3.$$

It sends k_{LR} to the generator of the Pati–Salam $\mathbb{Z}_2$ kernel. Thus its full inverse image is $\langle k_{\text{LR}} \rangle$, and quotienting gives the injective lower arrow $G_{\text{LR}} \hookrightarrow G_{\text{PS}}$. The first lifted inclusion sends the Standard Model kernel generator to k_{LR} , so the charge lattice is compatible along the chain.

Remark (Interchange symmetry over a fixed (D, Θ)). The 1,080 pairs (D_λ, Θ) form one Weyl orbit. By the injectivity of $\lambda \mapsto D_\lambda$, every element stabilizing the pair fixes λ , so the context stabilizer lies in $W(D_\lambda) \cong W(D_5)$. If $R = D_\lambda \cap \Theta \cong A_2 + A_1 + A_1$, a direct normalizer calculation gives

$$\text{Stab}_{\mathcal{W}}(D_\lambda, \Theta) = N_{W(D_5)}(R) \cong W(R) \rtimes \mathbb{Z}_2, \quad |\text{Stab}| = 48.$$

The nontrivial factor exchanges the two A_1 -components, equivalently the two distinguished root pairs $\{\pm\rho_1\}$ and $\{\pm\rho_2\}$. It fixes the common-maximal root system

$$C_3 + \{\pm\rho_1\} + \{\pm\rho_2\}$$

and exchanges the two pairwise-maximal root systems $C_3 + \{\pm\rho_1\}$ and $C_3 + \{\pm\rho_2\}$. This is an internal Weyl symmetry of the subgroup configuration, distinct from any spacetime parity operation.

5 The 27 and the Standard Model

We restrict $V = \mathbf{27} \boxtimes \mathbf{3}$ along the faithful Standard Model chain. First we give its complete branching, then determine the charge lattice and compute the net chiral class. The three-dimensional multiplicity factor gives three copies of every summand in the restriction of one **27**.

5.1 One 27: a chiral family, a vectorlike pair, and two singlets

Restriction along $E_6^{\text{sc}} \supset \text{Spin}(10) \supset SU(5)$ gives

$$\begin{aligned} \mathbf{27} \downarrow \text{Spin}(10) &= \mathbf{16} \oplus \mathbf{10} \oplus \mathbf{1} \\ \downarrow SU(5) &= \underbrace{(\mathbf{10} \oplus \bar{\mathbf{5}})}_{\mathcal{F}_{\text{SM}}} \oplus (\mathbf{5} \oplus \bar{\mathbf{5}}) \oplus 2\mathbf{1}. \end{aligned} \tag{8}$$

The restriction $\mathcal{F}_{\text{SM}} = (\mathbf{10} \oplus \bar{\mathbf{5}})|_{G_{\text{SM}}}$ is $(\mathbf{3}, \mathbf{2})_{1/6} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-2/3} \oplus (\mathbf{1}, \mathbf{1})_1 \oplus (\bar{\mathbf{3}}, \mathbf{1})_{1/3} \oplus (\mathbf{1}, \mathbf{2})_{-1/2}$, the fermion representation of one minimal Standard Model family in an all-left-handed convention [14, 15]. Denote these five summands by Q, u^c, e^c, d^c, L , respectively. The G_{SM} -singlet $\nu^c \subset \mathbf{16}$ is one of the two singlets in (8), not a summand of $\mathcal{F}_{\text{SM}}$. Equations (7)–(8) give the full decomposition

$$V|_{G_{\text{SM}}} \cong 3\mathcal{F}_{\text{SM}} \oplus 3(\mathbf{5}|_{G_{\text{SM}}} \oplus \bar{\mathbf{5}}|_{G_{\text{SM}}}) \oplus 6\mathbf{1}. \tag{9}$$

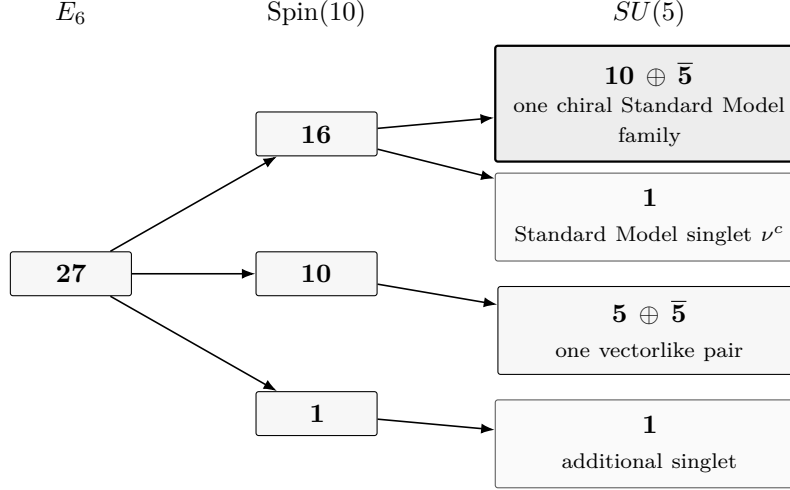


Figure 2: One **27** branches into a chiral Standard Model family, a vectorlike $\mathbf{5} \oplus \bar{\mathbf{5}}$ pair, and two singlets. The factor $\mathbf{3}_M$ gives three copies of this branching.

The six singlets consist of three copies of the singlet in the **16** of Spin(10) and three copies of the additional singlet in the **27** of E_6 .

5.2 The faithful Standard Model group and its charge lattice

Corollary 4.8 gives one E_8 -conjugacy class, so the restriction may be computed in the block representative

$$G_{\text{SM}} = S(U(3) \times U(2)) \subset SU(5) \subset \text{Spin}(10) \subset E_6^{\text{sc}}.$$

Its central circle has the primitive parametrisation, up to inversion,

$$\eta(z) = \text{diag}(z^{-2}I_3, z^3I_2).$$

This circle homomorphism is a *cocharacter*; primitive means that it traverses the circle once. Orient η so that the $(\mathbf{3}, \mathbf{2})$ in the **10** has integer charge $q = 1$. The conventional hypercharge used above is then $Y = q/6$. Let $t \in \mathbb{Z}/3$ be colour triality and $s \in \mathbb{Z}/2$ weak-centre parity: the central elements $e^{2\pi i/3}I_3 \in SU(3)$ and $-I_2 \in SU(2)$ act as $e^{2\pi i t/3}$ and $(-1)^s$, respectively. The kernel in Corollary 4.8 says that a representation of integer charge q descends to G_{SM} exactly when

$$q + 2t + 3s \equiv 0 \pmod{6}.$$

This congruence is the $\mathbb{Z}_6$ charge lattice. When $t = 0$, it links q to the $SU(2)$ centre parity and forces every electric charge in a colour-singlet multiplet to be integral.

5.3 The net chiral index is three

Net chirality records the difference between the multiplicities of an irreducible representation and its dual. A dual pair contributes zero, as does a self-dual summand. The following quotient records all these differences at once.

Let $R(G_{\text{SM}})$ be the additive Grothendieck group of finite-dimensional complex representations. Let χ be the quotient map obtained by imposing $\chi(R^*) = -\chi(R)$ and $\chi(S) = 0$ for every self-dual S .

The map χ records, for each non-self-dual pair $\{R, R^*\}$, the multiplicity difference $m_R - m_{R^*}$. Applying it to (9) gives

$$\begin{aligned}\chi(V|_{G_{\text{SM}}}) &= 3\chi(\mathcal{F}_{\text{SM}}) + 3(\chi(\mathbf{5}|_{G_{\text{SM}}}) + \chi(\bar{\mathbf{5}}|_{G_{\text{SM}}})) + 6\chi(\mathbf{1}) \\ &= \boxed{3\chi(\mathcal{F}_{\text{SM}})}.\end{aligned}\tag{10}$$

The vectorlike pairs and singlets make no contribution. Equation (10) records net chirality; equation (9) records the full module. Duality gives

$$\{\chi(V|_{G_{\text{SM}}}), \chi(V^*|_{G_{\text{SM}}})\} = \{3\chi(\mathcal{F}_{\text{SM}}), -3\chi(\mathcal{F}_{\text{SM}})\}.$$

Relative to $\chi(\mathcal{F}_{\text{SM}})$, the coefficients $+3$ and -3 are the chiral indices of V and V^* , respectively.

Corollary 5.1 (Chiral content of the classified module). *The two orientations have chiral-index coefficients $+3$ and -3 , relative to $\chi(\mathcal{F}_{\text{SM}})$. Beyond this family class, no additional net chiral class occurs in V or V^* ; every remaining summand is vectorlike or a singlet.*

5.4 Local anomalies and the $SU(3)_M^3$ background anomaly

Standard Model insertions. Perturbative local cubic and linear anomaly coefficients are additive, odd under duality, and zero on self-dual modules, so they factor through χ . The linear coefficient governs the mixed gauge–gravitational anomaly. Since $\text{Lie}(G_{\text{SM}}) \subset \mathfrak{e}_6 = \text{Rad}(c_V)$, the cubic anomaly of $V|_{G_{\text{SM}}}$ vanishes. The algebra $\mathfrak{e}_6$ is perfect, so every finite-dimensional trace functional on it also vanishes. Equation (10) then shows that one copy of $\mathcal{F}_{\text{SM}}$, and hence three copies, is free of perturbative gauge and mixed gauge–gravitational anomalies.

The preceding argument covers perturbative local anomalies. The spectrum in (9) also has no mod-two $SU(2)$ anomaly: counting colour multiplicity, each $\mathcal{F}_{\text{SM}}$ contains four doublets, while each vectorlike $\mathbf{5} \oplus \bar{\mathbf{5}}$ contributes two. The same spectrum is the restriction of anomaly-free $SU(5)$ representations. For theories on spin manifolds, the bordism analysis of the four connected Standard Model global forms gives no further global gauge anomaly for the $\mathbb{Z}_6$ quotient used here [26].

The family background. To identify the remaining cubic anomaly, let F_{SM} be a G_{SM} -gauge field strength and F_M a background field strength for $SU(3)_M$. On $V = \mathbf{27} \otimes \mathbf{3}_M$, put

$$F = F_{\text{SM}} \otimes \mathbf{13} + \mathbf{127} \otimes F_M.$$

The radical condition kills every cubic term containing F_{SM} , and both simple factors act tracelessly. Therefore

$$\text{Tr}_V(F^3) = 27 \text{Tr}_{\mathbf{3}}(F_M^3), \quad \text{Tr}_V(F) = 0.\tag{11}$$

Every perturbative local anomaly coefficient with a G_{SM} gauge insertion vanishes, including coefficients mixed with the background $SU(3)_M$ field. The only remaining cubic component is the $SU(3)_M^3$ term, with coefficient 27 when $A(\mathbf{3}) = 1$. As a background global symmetry, the commuting $SU(3)_M$ carries this perturbative $SU(3)_M^3$ anomaly. Promoting it to a gauge factor requires additional anomaly-cancelling structure.

6 Conclusion

The classification starts with an effective inclusion of compact Lie algebras and its adjoint complement. Complex type and a nonzero cubic radical determine the pair. Up to isomorphism of Lie-algebra pairs and duality, the unique solution is

$$\mathfrak{e}_8 \supset \mathfrak{e}_6 \oplus \mathfrak{su}(3)_M, \quad V = \mathbf{27} \boxtimes \mathbf{3}, \quad \text{Rad}(c_V) = \mathfrak{e}_6.$$

Every inequivalent pair satisfying the standing algebraic hypotheses and $\text{End}_{\mathfrak{h}}(\mathfrak{q}) \cong \mathbb{C}$ has zero cubic radical. Thus no nonzero ideal $\mathfrak{k}$ satisfies $c_V(\mathfrak{k}, \mathfrak{h}, \mathfrak{h}) = 0$. This obstruction depends only on the embedding and its quotient module.

The classification covers effective proper inclusions of compact real Lie algebras of arbitrary rank. Neither algebra type, representation dimension, nor multiplicity is prescribed. Irreducibility of $\mathfrak{q}$, simplicity of $\mathfrak{g}$, and semisimplicity and maximality of $\mathfrak{h}$ follow from the conditions. They also give a compact homogeneous realization with closed subgroup; homogeneity expresses the geometry of the pair rather than an extra selection rule. The finite searches arise from proved reductions and bounds; the classical lower-rank exclusion is uniform in rank. Higher rank supplies no alternative.

Unlike a construction around a chosen group or coset, this theorem varies the embedding and its canonical representation together. This also distinguishes it from classifications of semisimple extensions of a supplied gauge algebra and spectrum [27], and from studies of how often anomaly-free spectra admit unification [28]. The familiar exceptional pair is characterized, not assumed.

Inside its E_6 factor, the second classification gives six Weyl orbits of maximal-subsystem triples, exactly two with a common A_2 component. In every triple, the unique largest pairwise derived group is obtained from the $D_5-(A_5 + A_1)$ pair. Using that group in the pair-first construction gives the Standard Model and left-right chains on the two extremal orbits, with their faithful global forms and hypercharge embedding. The chiral dimensions 15 and 16 follow from restriction to the intermediate groups. The full branching (9) and chiral class (10) distinguish three minimal families from the additional vectorlike pairs and singlets.

Ordinary matter is built from electrons and the lightest quarks. Their heavier relatives carry the same gauge charges, repeating the family pattern three times. The classified module contains that same threefold chiral family class. Here the coefficient and the cubic-anomaly-safe ideal are not chosen independently: they follow from one embedding.

Computational material

The repository linked on the title page contains the exact programs, certificates and master verification notebook [29]. The finite algorithms and completeness arguments used by the proof are stated in Sections 3–4; the notebook supplies the reproducible calculations and output tables.

The repository also contains Lean checks of the finite exceptional grading calculation, the Cartan cubic test, and the stated E_6 incidence propositions. These checks use `native_decide` with the compiler trust described in the repository. The manuscript supplies the compact-Lie reductions, the classical all-rank argument and the representation-theoretic proofs around these finite checks.

AI assistance disclosure. This paper was created with the aid of the Intelligent Internet Zenith system. The paper was checked with GPT-6 Astra Pro and Fable 5.1 Max. The repository and computational verification were generated by GPT-6 Astra.

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